Painlevé IV asymptotics for orthogonal polynomials with respect to a modified Laguerre weight

D. Dai and A.B.J. Kuijlaars*

Department of Mathematics, Katholieke Universiteit Leuven,

Celestijnenlaan 200 B, 3001 Leuven, Belgium.

dan.dai@wis.kuleuven.be, arno.kuijlaars@wis.kuleuven.be

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Abstract

We study polynomials that are orthogonal with respect to the modified Laguerre weight $z^{-n+\nu}e^{-Nz}(z-1)^{2b}$ in the limit where $n, N \to \infty$ with $N/n \to 1$ and ν is a fixed number in $\mathbb{R} \setminus \mathbb{N}_0$. With the effect of the factor $(z-1)^{2b}$, the local parametrix near the critical point $z = 1$ can be constructed in terms of Ψ-functions associated with the Painlevé IV equation. We show that the asymptotics of the recurrence coefficients of orthogonal polynomials can be described in terms of specified solution of the Painlevé IV equation in the double scaling limit. Our method is based on the Deift/Zhou steepest decent analysis of the Riemann-Hilbert problem associated with orthogonal polynomials.

Key words: Painlevé IV equation, Riemann-Hilbert problem, Deift/Zhou steepest decent analysis.

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1 Introduction and statement of results

1.1 Modified Laguerre weights

Let α, b be real constants, and let N be a positive number. Define the (complex-valued) weight function to be

$$w(z) = z^{\alpha} e^{-Nz} (z-1)^{2b}, \tag{1.1}$$

where z^{α} and $(z-1)^{2b}$ are defined with cuts along $[0,\infty)$ and $[1,\infty)$, respectively. That is, $z^{\alpha} = |z|^{\alpha} e^{i\alpha \arg z}$ with $0 < \arg z < 2\pi$, and $(z-1)^{2b} = |z-1|^{2b} e^{2ib \arg(z-1)}$ with $0 < \arg(z-1) < 2\pi$. Let Σ be a contour in $\mathbb{C} \setminus [0,\infty)$ that is symmetric with respect to the real axis, tends to infinity in the horizontal direction and remains bounded in the vertical direction. This contour divides the complex plane into two domains $\Omega_{\pm}$; see Figure 1. We consider monic polynomials π_n of degree n, $\pi_n(z) = z^n + \cdots$, that are orthogonal with respect to $w(z)$ on Σ, i.e.,

$$\int_{\Sigma} \pi_n(z) z^j w(z) dz = 0, \qquad j = 0, 1, \ldots, n-1. \tag{1.2}$$

This is an example of non-Hermitian orthogonality, for which there is no general existence and uniqueness result associated with such $w(z)$. It will be part of our results that in the asymptotic regime that we will consider, the polynomial π_n uniquely exists for n large enough.

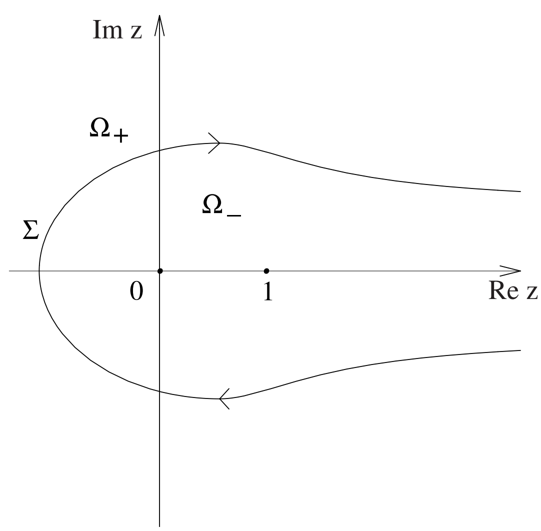

Figure 1: The contour Σ

For $b = 0$ and $\alpha > -1$ (with $\alpha \notin \mathbb{Z}$) we may deform the contour of integration to $[0, \infty)$. Then the orthogonality condition (1.2) reduces to

$$\int_0^\infty \pi_n(x) x^{j+\alpha} e^{-Nx} dx = 0, \qquad j = 0, 1, \ldots, n-1, \tag{1.3}$$

which shows that in this case π_n is related to the classical Laguerre polynomial with parameter α,

$$\pi_n(z) = (-1)^n \frac{n!}{N^n} L_n^{(\alpha)}(Nz). \tag{1.4}$$

For properties of the classical Laguerre polynomial $L_n^{(\alpha)}(z)$, see e.g. Abramowitz and Stegun [1] or Szegő [2]. For $b = 0$ and $\alpha < -1$, the orthogonality condition (1.3) is not valid, but the relation (1.4) still holds true. Asymptotic properties of the Laguerre polynomials with large negative parameters were studied by Kuijlaars and McLaughlin [3, 4] in the regime where $N = n$, $n \to \infty$, $\alpha \to -\infty$ in such a way that

$$\lim_{n \to \infty} -\frac{\alpha}{n} = A$$

exists. Then it was found that the zeros of π_n accumulate either on an open contour (if $A > 1$) or on a union of a closed contour and a real interval (if $0 < A < 1$); see Figure 2. For the special value $A = 1$, the zeros typically (but not always!) accumulate on the Szegő curve $\mathcal{S}$ defined by

$$\mathcal{S} := \{z \in \mathbb{C} : |ze^{1-z}| = 1 \quad \text{and} \quad |z| \leq 1\}. \tag{1.5}$$

In the special case that $\alpha = -n + \nu$ with $\nu \notin \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ fixed, it was also found [5, 6] that the local asymptotics of the polynomials π_n of (1.4) near the point $z = 1$ are given in terms of parabolic cylinder functions $D_\nu(z)$. The main tool in the analysis of [3, 4, 5, 6] is the matrix valued Riemann-Hilbert (RH) problem for Laguerre polynomials and the Deift/Zhou method of steepest descent, which was introduced in [7] and first applied to orthogonal polynomials in [8, 9]; see also [10]. The parabolic cylinder functions appear via a construction of a local parametrix in a neighborhood of the point $z = 1$.

The main effect of the extra factor $(z-1)^{2b}$ in the weight (1.1) is in the local behavior of the polynomials near the point $z = 1$. From the point of view of Riemann-Hilbert analysis it means that in the critical case $A = 1$ the construction of the local parametrix with parabolic cylinder functions will no longer work. It is the aim of this paper to show that in this situation the role of the parabolic cylinder functions is replaced by the Ψ functions associated with a special solution of the Painlevé IV equation. To the best of our knowledge, this is the first time that Painlevé IV

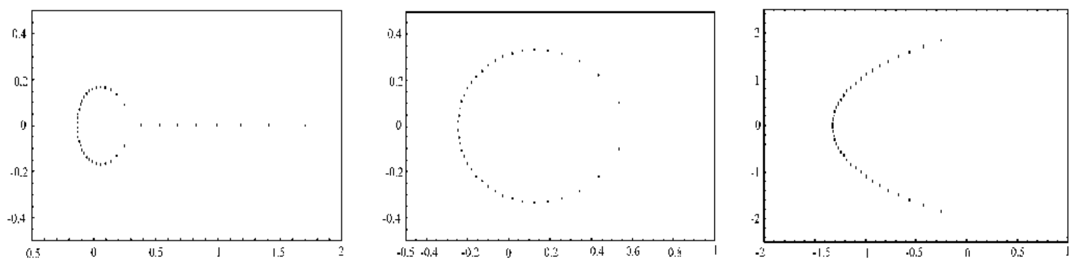

Figure 2: Zeros of $L_n^{(-nA)}(nz)$ for $n = 40$, and $A = 0.81$ (left), $A = 1.001$ (middle), and $A = 2$ (right).

functions are used in the construction of a local parametrix in a RH steepest descent analysis.

We were guided in our approach by a number of recent advances in random matrix theory and the theory of orthogonal polynomials. In critical situations it was found that Painlevé transcendents appear in the description of the local eigenvalue statistics of random matrices as well as in the local asymptotics of orthogonal polynomials near singular points. For example, the local eigenvalue statistics at the opening of a gap in the spectrum of unitary invariant random matrix ensembles is described by the Hastings-McLeod solution of the Painlevé II equation [11, 12, 13]. The same function is well-known to play a role in the Tracy-Widom distributions for the largest eigenvalue of random matrices [14] and the problem of the length of the longest increasing subsequence of a random permutation [15]. Singular behavior at edge points may be described by special solutions of the Painlevé I equation [16, 17], the second member of the Painlevé I hierarchy [18, 19], and the Painlevé XXXIV equation [20]. The Riemann-Hilbert approach was used in all of these situations.

In the statement of the results in this paper we will focus on the behavior of the recurrence coefficients in the three-term recurrence relation

$$\pi_{n+1}(z) = (z - b_n)\pi_n(z) - a_n \pi_{n-1}(z) \tag{1.6}$$

in the asymptotic regime where $n \to \infty$ with

$$\alpha = -n + \nu, \qquad N = n + \sqrt{2}\, L n^{1/2} \tag{1.7}$$

with b, ν and L are real constants and independent of n. Then of course the recurrence coefficients depend on the constants b, ν and L, but we suppress it in our notation. In this regime it turns out that we see the appearance of the Painlevé IV transcendent.

As noted before, for $b = 0$ the polynomials reduce to rescaled Laguerre polynomials (1.4). From the recurrence relation for Laguerre polynomials

$$-zL_n^{(\alpha)}(z) = (n+1)L_{n+1}^{(\alpha)}(z) - (2n+\alpha+1)L_n^{(\alpha)}(z) + (n+\alpha)L_{n-1}^{(\alpha)}(z) \tag{1.8}$$

and (1.4) it follows that under the scaling (1.7) the recurrence coefficients behave as

$$a_n = \frac{\nu}{n} + O(n^{-3/2}), \qquad b_n = 1 - \frac{\sqrt{2}\,L}{n^{1/2}} + O(n^{-1}), \qquad \text{as } n \to \infty. \tag{1.9}$$

For general $b \in \mathbb{R}$ and $\nu \notin \mathbb{N}_0$, we prove that (except for a discrete set of values L) a_n and b_n exist for n large enough, and satisfy

$$a_n = \frac{a^{(1)}}{n} + O(n^{-3/2}), \qquad b_n = 1 - \frac{b^{(1)}}{n^{1/2}} + O(n^{-1}),$$

for certain explicit constants $a^{(1)}$ and $b^{(1)}$, which depend on b, ν and L, that are explicitly calculated in terms of functions associated with the Painlevé IV equation. For $b = 0$ they reduce to $a^{(1)} = \nu$ and $b^{(1)} = \sqrt{2}\,L$.

Before we can state our result we first discuss the Painlevé IV equation.

1.2 Painlevé IV equation

The Painlevé IV (PIV) equation is defined as

$$\frac{d^2u}{ds^2} = \frac{1}{2u}\left(\frac{du}{ds}\right)^2 + \frac{3}{2}u^3 + 4su^2 + 2(s^2 + 1 - 2\Theta_\infty)u - \frac{8\Theta^2}{u}, \tag{1.10}$$

where Θ and Θ_∞ are constants. It is known that all solutions $u(s)$ of PIV are meromorphic functions in the complex s-plane and all poles of $u(s)$ are simple with the residue ± 1; see [21, 22, 23]. For more properties of the PIV transcendent, see a very good review article by Clarkson [24] and references therein. The Ψ-functions associated with Painlevé IV are solutions of the following system of linear differential equations (Lax pair) for $\Psi(l, s)$ given first by Jimbo and Miwa [25]

$$\frac{\partial \Psi}{\partial l} = A\Psi, \quad \frac{\partial \Psi}{\partial s} = B\Psi, \tag{1.11}$$

with

$$\begin{aligned} A = &\left(l + s + \frac{1}{l}(\Theta - K)\right)\sigma_3 + y\left(1 - \frac{u}{2l}\right)\sigma_+ \\ &+ \frac{2}{y}\left(K - \Theta - \Theta_\infty + \frac{K}{lu}(K - 2\Theta)\right)\sigma_-, \end{aligned} \tag{1.12}$$

$$B = l\sigma_3 + y\sigma_+ + \frac{2}{y}(K - \Theta - \Theta_\infty)\sigma_-, \tag{1.13}$$

where $y = y(s)$ and $K = K(s)$ are defined by

$$\frac{1}{y}\frac{dy}{ds} = -u - 2s, \tag{1.14}$$

$$K = \frac{1}{4}\Big(-\frac{du}{ds} + u^2 + 2s\,u + 4\Theta \Big) \tag{1.15}$$

and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1.16}$$

The Painlevé IV equation (1.10) is the compatibility condition for the system (1.11). It should be noted that the Lax pair is not unique. For example, one can find another Lax pair in Kitaev [26] and Milne, Clarkson and Bassom [27].

To explain the special solutions of PIV that will appear in our results, we recall the RH problem for $\Psi(l, s)$ associated with PIV. In the RH problem $\Psi(\lambda, s)$ is viewed as

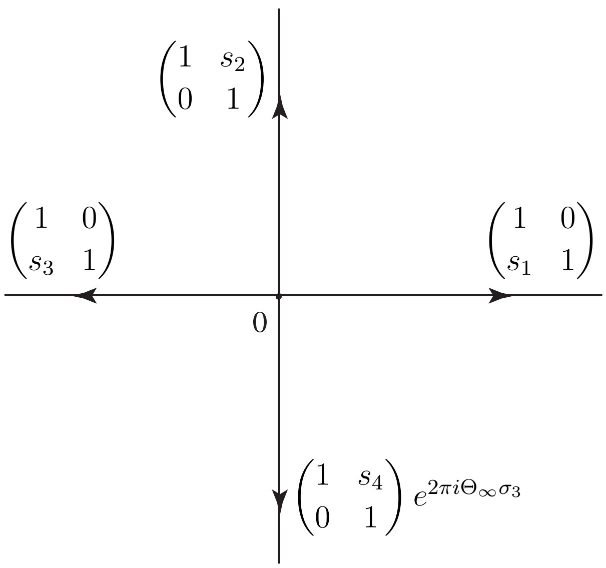

Figure 3: The contour and the jump matrices for the RH problem for Ψ

a function of λ, with s appearing as a parameter in the asymptotic condition (1.23). The constants Θ_∞ and Θ appear in the behaviors at infinity (1.23) and at the origin (1.24), respectively. The jump matrices in the RH problem include four parameters (Stokes multipliers) s_j, $j = 1, \ldots, 4$, satisfying

$$(1 + s_2 s_3) e^{2\pi i\,\Theta_\infty} + [s_1 s_4 + (1 + s_3 s_4)(1 + s_1 s_2)] e^{-2\pi i\,\Theta_\infty} = 2\cos 2\pi\Theta. \tag{1.17}$$

The RH problem for Ψ is (see [28, Section 5.1] and [29])

(a) $\Psi(l,s)$ is analytic for $l \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$; see Figure 3;

(b) on the contours $\mathbb{R} \cup i\,\mathbb{R}$ (with orientation as in Figure 3),

$$\Psi_+(l,s) = \Psi_-(l,s)\, S_1, \qquad \text{for } l \in \mathbb{R}_+; \tag{1.18}$$

$$\Psi_+(l,s) = \Psi_-(l,s)\, S_2, \qquad \text{for } l \in i\,\mathbb{R}_+; \tag{1.19}$$

$$\Psi_+(l,s) = \Psi_-(l,s)\, S_3, \qquad \text{for } l \in \mathbb{R}_-; \tag{1.20}$$

$$\Psi_+(l,s) = \Psi_-(l,s)\, S_4\, e^{2\pi i \Theta_\infty \sigma_3}, \quad \text{for } l \in i\,\mathbb{R}_-, \tag{1.21}$$

where

$$\begin{aligned} S_1 &= \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix}, \qquad & S_2 &= \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix}, \\ S_3 &= \begin{pmatrix} 1 & 0 \\ s_3 & 1 \end{pmatrix}, \qquad & S_4 &= \begin{pmatrix} 1 & s_4 \\ 0 & 1 \end{pmatrix}; \end{aligned} \tag{1.22}$$

(c) as $l \to \infty$:

$$\Psi(l,s) = \left(I + \frac{\Psi_{-1}(s)}{l} + \frac{\Psi_{-2}(s)}{l^2} + O\left(\frac{1}{l^3}\right)\right) e^{(\frac{l^2}{2}+sl)\sigma_3}\; l^{-\Theta_\infty \sigma_3}, \tag{1.23}$$

where $l^{-\Theta_\infty}$ is defined with a branch cut along $i\,\mathbb{R}_-$;

(d) as $l \to 0$:

$$\Psi(l,s)\mathcal{C}^{-1} l^{-\Theta\sigma_3} = O(1) \qquad \text{for } \operatorname{Re} l > 0 \text{ and } \operatorname{Im} l < 0. \tag{1.24}$$

where l^{Θ} is defined with a branch cut along $i\,\mathbb{R}_-$ and the connection matrix $\mathcal{C}$ is any invertible matrix satisfying

$$S_1 S_2 S_3 S_4 e^{2\pi i \Theta_\infty \sigma_3} = \mathcal{C}^{-1} e^{-2\pi i \Theta \sigma_3} \mathcal{C}. \tag{1.25}$$

Then, from Theorem 5.2 in [28], we obtain a solution of PIV (1.10) with constants Θ and Θ_∞, by putting

$$u(s) := -2s - \frac{d}{ds} \log \Big((\Psi_{-1})(s) \Big)_{12}. \tag{1.26}$$

The solution depends on the constants s_j's in (1.22), which as explained should satisfy the equation (1.17). The equation (1.17) is invariant under the transformation

$$\{s_1, s_2, s_3, s_4\} \mapsto \{ds_1, d^{-1}s_2, ds_3, d^{-1}s_4\}, \qquad d \neq 0. \tag{1.27}$$

The Ψ function then transforms as $\Psi \mapsto d^{-\sigma_3/2}\Psi d^{\sigma_3/2}$ which leaves the solution (1.26) invariant under the transformation (1.27). The connection matrix $\mathcal{C}$ transforms as $\mathcal{C} \mapsto d^{-\sigma_3/2}\mathcal{C}\, d^{\sigma_3/2}$. The PIV solution (1.26) is a meromorphic function of s and the RH problem for Ψ has a solution if and only if s is not a pole of u.

The special solution of (1.10) that will appear in our results corresponds to the special Stokes multipliers

$$\begin{aligned} s_1 &= (e^{(\Theta_\infty-\Theta)\pi i} - e^{-(\Theta_\infty-\Theta)\pi i})e^{\Theta_\infty \pi i}, & s_2 &= e^{-\Theta\pi i}, \\ s_3 &= -(e^{(\Theta_\infty+\Theta)\pi i} - e^{-(\Theta_\infty+\Theta)\pi i})e^{-\Theta_\infty \pi i}, & s_4 &= -e^{(2\Theta_\infty+\Theta)\pi i}. \end{aligned} \tag{1.28}$$

where

$$\Theta = -b \quad \text{and} \quad \Theta_\infty = \nu + b. \tag{1.29}$$

It is easy to check that (1.28) indeed satisfies the relation (1.17). Furthermore, by (1.25), the corresponding connection matrix $\mathcal{C}$ is

$$\mathcal{C} = \begin{pmatrix} r_1 & 0 \\ 0 & r_1^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e^{(2\Theta_\infty-\Theta)\pi i} & 1 \end{pmatrix} \begin{pmatrix} 1 & r_2 \\ 0 & 1 \end{pmatrix}, \quad r_1 \neq 0$$

and r_1 and r_2 are arbitrary constants satisfying $r_2 = 0$, if $b = m/2, m \in \mathbb{Z}$. Throughout this paper, we will choose $r_1 \equiv 1$ and $r_2 \equiv 0$, which means

$$\mathcal{C} = \begin{pmatrix} 1 & 0 \\ -e^{(2\Theta_\infty-\Theta)\pi i} & 1 \end{pmatrix}, \tag{1.30}$$

or after a transformation (1.27)

$$\mathcal{C} = \begin{pmatrix} 1 & 0 \\ -de^{(2\Theta_\infty-\Theta)\pi i} & 1 \end{pmatrix}. \tag{1.31}$$

1.3 Statement of results

The asymptotic formulas for the recurrence coefficients a_n and b_n in (1.6) are given as follows:

Theorem 1.1. *Suppose $b \in \mathbb{R}$, $\nu \in \mathbb{R} \setminus \mathbb{N}_0$ and put*

$$\Theta = -b, \qquad \Theta_\infty = \nu + b. \tag{1.32}$$

Let $u(s)$ be the special solution of PIV corresponding to the Stokes multipliers (1.28) *and let $K(s)$ be defined in (1.15). Assume*

$$\alpha = -n + \nu, \qquad N = n + \sqrt{2}\, Ln^{1/2} \tag{1.33}$$

where L is not a pole of $u(s)$. Then, for every large enough n, the monic polynomial π_n of degree n, satisfying

$$\int_\Sigma \pi_n(z) z^{j-\alpha} e^{-Nz}(z-1)^{2b} dz = 0, \qquad j = 0, 1, \ldots, n-1,$$

uniquely exists. The polynomials satisfy the recurrence relation (1.6) *with recurrence coefficients a_n and b_n satisfying*

$$a_n = \frac{1}{n}\Big(\nu - K(L)\Big) + O(n^{-3/2}) \qquad \text{as } n \to \infty \tag{1.34}$$

and, if $K(L) \neq \nu$,

$$b_n = 1 + \frac{\sqrt{2}}{\sqrt{n}}\left[\frac{K(L)\Big(K(L)+2b\Big)}{u(L)\Big(K(L)-\nu\Big)} - L\right] + O(n^{-1}) \qquad \text{as } n \to \infty. \tag{1.35}$$

In the Theorem 1.1, (1.35) still makes sense when $u(L) = 0$. In fact, from the PIV equation in (1.10), it can be verified that

$$u'(L) = \pm 4\Theta = \mp 4b \qquad \text{if} \quad u(L) = 0.$$

From (1.15), then we have $K(L) = 0$ or $K(L) = 2\Theta = -2b$. Then by L'Hospital's rule, we get from (1.35)

$$b_n = 1 + \tfrac{\sqrt{2}}{\sqrt{n}}\left[\tfrac{K'(L)}{2\nu} - L\right] + O(n^{-1}), \qquad \text{if } u(L) = 0 \text{ and } u'(L) = -4b, \tag{1.36}$$

$$b_n = 1 + \tfrac{\sqrt{2}}{\sqrt{n}}\left[\tfrac{K'(L)}{2(\nu+2b)} - L\right] + O(n^{-1}), \qquad \text{if } u(L) = 0 \text{ and } u'(L) = 4b. \tag{1.37}$$

Note that the factor $(\nu + 2b)$ in (1.37) can not be zero because of the assumption $K(L) \neq \nu$ in (1.35).

When L is not a pole of $u(s)$, it is still possible that $K(L) = \nu$. In this case, the RH problem for Ψ in (1.18)–(1.24) is solvable. Moreover, for large enough n, the monic polynomial π_n still uniquely exists. However something strange occurs for the recurrence coefficient b_n. Note that (1.35) no longer holds when $K(L) = \nu$. In fact, as $n \to \infty$, b_n does not tend to 1 but to another constant, which means there is a sudden change from $K(L) \neq \nu$ to $K(L) = \nu$. At this time, we can not explain why this phenomenon happens.

In [30], Murata found the following Schlesinger transformation

$$u^*(s) = -\frac{2\,K(s)\Big(K(s)+2b\Big)}{u(s)\Big(K(s)-\nu\Big)}, \tag{1.38}$$

where $u^*(s)$ is a solution to the PIV equation with parameters $\Theta = -b, \Theta_\infty = \nu+b+1$. Then, with (1.38), one can rewrite (1.35) as

$$b_n = 1 - \frac{\sqrt{2}}{\sqrt{n}} \left(\frac{1}{2} u^*(L) + L \right) + O(n^{-1}), \qquad n \to \infty, \tag{1.39}$$

whenever L is not a pole of $u^*(s)$.

The case $\nu \in \mathbb{N}_0$ is special and we do not understand it at this moment. That this is a particular situation can already be seen in the Laguerre case treated in [4], since it corresponds to Laguerre polynomials with a zero of large order at the origin, and the RH problem stated in Section 2 below does not have a solution in case $\nu \in \mathbb{N}_0$ and $n > \nu$, see Proposition 2.2 of [3].

Theorem 1.1 could still be true in case $\nu \in \mathbb{N}_0$, but our method of proof fails. Indeed, in the transformation (3.33) we make essential use of the fact that ν is not an integer. Therefore we exclude the case $\nu \in \mathbb{N}_0$ from our considerations.

Using the Riemann-Hilbert method, it is possible to obtain asymptotic expansion for polynomials $\pi_n(z)$ in all regions of the full complex z-plane as $n \to \infty$. Near the point $z = 1$, the expansion involves the Ψ-functions given in (1.11). We will not discuss it in this paper.

For the polynomials, we restrict ourselves to the asymptotic zero distribution.

Theorem 1.2. *Under the same assumptions as in Theorem 1.1 and $K(L) \neq \nu$, we have that all the zeros of polynomials $\pi_n(z)$ tend to the Szegő curve $\mathcal{S}$ given in* (1.5). *More precisely, for any neighborhood $U(\mathcal{S})$ of $\mathcal{S}$, there exists a positive integer n_0 such that for any $n > n_0$, all zeros of $\pi_n(z)$ lie in $U(\mathcal{S})$.*

Outline of the rest of the paper

In Section 2, we formulate the RH problem for orthogonal polynomials π_n, which is the starting point of the asymptotic analysis. We have to distinguish the two cases

Case I: $\nu > 0$ and $\nu \notin \mathbb{N}$,

Case II: $\nu < 0$,

since the steepest descent analysis is different for the two cases. The two cases are dealt with in Sections 3 and 4 respectively. Together they contain the proof of Theorem 1.1. In Section 5, we prove Theorem 1.2. Finally in Section 6, we show that, with certain parameters Θ and Θ_∞, the special solutions of PIV studied in this paper are given in terms of parabolic cylinder functions.

Since the two cases Case I and Case II are independent of each other, we are going to use the same notation for the functions and variables. We trust that this will not lead to any confusion.

2 The RH problem for orthogonal polynomials

Consider the following RH problem for a 2×2 matrix valued function $Y : \mathbb{C} \setminus \Sigma \mapsto \mathbb{C}^{2\times 2}$:

(a) $Y(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma$; see Figure 1. Here Σ is an oriented curve whose $+$-side and $-$-side are on the left and right while traversing the contour, respectively,

(b) $Y(z)$ possesses continuous boundary values $Y_\pm(z)$ such that

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & w(z) \\ 0 & 1 \end{pmatrix} \quad \text{for } z \in \Sigma. \tag{2.1}$$

(c) $Y(z)$ has the following behavior as $z \to \infty$

$$Y(z) = \left(I + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}.$$

By a by now standard argument, originally due to Fokas, Its, and Kitaev [17], the solution of the above RH problem, if it exists, is uniquely given by

$$Y(z) = \begin{pmatrix} \pi_n(z) & \dfrac{1}{2\pi i}\displaystyle\int_\Sigma \frac{w(s)\,\pi_n(s)}{s-z} ds \\ p_{n-1}(z) & \dfrac{1}{2\pi i}\displaystyle\int_\Sigma \frac{w(s)\,p_{n-1}(s)}{s-z} ds \end{pmatrix}, \tag{2.2}$$

where π_n is a monic polynomial of degree n satisfying (1.2) and p_{n-1} is a polynomial of degree $\leq n-1$. The existence and uniqueness of the monic polynomial π_n satisfying (1.2) is equivalent to the solvability of the RH problem.

The recurrence coefficients a_n and b_n can be expressed in terms of the solution of the RH problem.

Proposition 2.1. *Assume that the RH problem has a solution Y and write*

$$Y(z) = \left(I + \frac{Y_{-1}}{z} + \frac{Y_{-2}}{z^2} + O(z^{-3})\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \tag{2.3}$$

as $z \to \infty$. Define

$$a_n = \left(Y_{-1}\right)_{12}\left(Y_{-1}\right)_{21} \tag{2.4}$$

and

$$b_n = \frac{\left(Y_{-2}\right)_{12}}{\left(Y_{-1}\right)_{12}} - \left(Y_{-1}\right)_{22}, \tag{2.5}$$

where the subscript ij denotes the (i,j) entry of the corresponding matrix. If $a_n \neq 0$, then the monic orthogonal polynomials π_{n+1}, π_n and π_{n-1} (of degrees $n+1$, n, and $n-1$) with respect to $w(z)$ on Σ, exist and they satisfy the recurrence relation

$$\pi_{n+1}(z) = \left(z - b_n\right)\pi_n(z) - a_n\, \pi_{n-1}(z). \tag{2.6}$$

Proof. The proof is standard, see [10]. For the above precise form, see also [16, Proposition 2.4]. ☐

As already said before, we are going to apply the Deift/Zhou steepest descent method to the above RH problem. In Sections 3 we treat the case $\nu > 0$ and in Section 4 the case $\nu < 0$. Before we embark on the steepest descent analysis let us say a few words about the method. The method consists of a number of explicit transformations, which in this paper take the form $Y \mapsto U \mapsto T \mapsto S \mapsto R$, which lead to a "simple" RH problem for R. Since the works of Deift et al. [8, 9, 10] it is now clear what the main steps should be in the analysis of the RH problem associated with orthogonal polynomials, namely

- normalization of the RH problem with the so-called g-function;
- opening of the lens around the oscillatory region;
- construction of global and local parametrices.

In any extension of the method (such as the one in this paper) these issues appear and it is unfortunate that all details have to be checked each time, since we do not have, and maybe cannot expect, a master theorem giving general conditions under which the method will work.

In the present paper we are dealing with orthogonality on a contour in the complex plane. To handle this feature we largely follow the steepest descent analysis done for Laguerre polynomials with large negative parameters in [4] (for $\nu > 0$) and in [3] (for $\nu < 0$). This will lead to an analysis on varying n-dependent curves depending on $A_n = 1 - \frac{\nu}{n}$. For $n \to \infty$ the curves tend to the Szegő curve. The precise analysis

is rather delicate near the critical point $z = 1$, since we have to arrive at a local RH problem that can be modelled by the RH problem associated with Painlevé IV. A simpler and maybe more natural approach would be to avoid the varying curves and work with the Szegő curve from the beginning. We tried to do this but we were not able to handle all difficulties this way.

3 Case I: $\nu > 0$ and $\nu \notin \mathbb{N}$

3.1 Introduction

We assume $\nu > 0$ and $\nu \notin \mathbb{N}$. Throughout we write

$$t := \frac{N}{n} = 1 + \sqrt{2}\, L n^{-1/2}. \tag{3.1}$$

Let

$$A_n := -\frac{\alpha}{n} = 1 - \frac{\nu}{n}. \tag{3.2}$$

Since we will concentrate on the asymptotics as $n \to \infty$, then without loss of generality throughout this paper we assume n is large enough. Thus, $A_n \in (0, 1)$. Following the Riemann-Hilbert steepest descent analysis in [4], we define

$$\beta_{1,n} := 2 - A_n - 2\sqrt{1 - A_n} = 1 + \frac{\nu}{n} - 2\sqrt{\frac{\nu}{n}}, \tag{3.3}$$

$$\beta_{2,n} := 2 - A_n + 2\sqrt{1 - A_n} = 1 + \frac{\nu}{n} + 2\sqrt{\frac{\nu}{n}} \tag{3.4}$$

and

$$R_n(z) := \sqrt{(z - \beta_{1,n})(z - \beta_{2,n})}, \quad z \in \mathbb{C} \setminus [\beta_{1,n}, \beta_{2,n}], \tag{3.5}$$

where $R_n(z) \sim z$ as $z \to \infty$. Let

$$\phi_n(z) = \frac{1}{2} \int_{\beta_{1,n}}^{z} \frac{R_n(s)}{s} ds, \qquad z \in \mathbb{C} \setminus ((-\infty, 0] \cup [\beta_{1,n}, \infty)), \tag{3.6}$$

where the path of integration from $\beta_{1,n}$ to z lies entirely in the region $\mathbb{C} \setminus ((-\infty, 0] \cup [\beta_{1,n}, \infty))$, except for the initial point $\beta_{1,n}$. The curves where $\operatorname{Re} \phi_n(z)$ is constant are trajectories of the quadratic differential (see Strebel [31])

$$-\frac{R_n(s)^2}{s^2} ds^2 = -\frac{(s - \beta_{1,n})(s - \beta_{2,n})}{s^2} ds^2, \tag{3.7}$$

which has two simple zeros at $\beta_{1,n}$ and $\beta_{2,n}$ and a double pole at 0. In [4] it is shown that there exists a unique curve $\Gamma_{0,n}$ as follows:

Definition 3.1. The contour $\Gamma_{0,n}$ is a simple closed curve encircling 0 once, so that

$$\operatorname{Re}\phi_n(z) = 0 \qquad \text{for} \quad z \in \Gamma_{0,n}; \tag{3.8}$$

see Figure 4.

The curve $\Gamma_{0,n}$ depends on n and it tends to the Szegő curve $\mathcal{S}$ (1.5) as $n \to \infty$.

Lemma 3.2. *As $n \to \infty$, the curve $\Gamma_{0,n}$ tends to the Szegő curve $\mathcal{S}$.*

Proof. From (3.3) and (3.4), we know that $\beta_{1,n}$ and $\beta_{2,n}$ both tend to 1 as $n \to \infty$. Then, by (3.5), $R_n(z) \to z - 1$. As a consequence,

$$\phi_n(z) \to \frac{1}{2}\int_1^z \frac{s-1}{s}\,ds = \frac{1}{2}(z - 1 - \log z) \quad \text{as } n \to \infty, \tag{3.9}$$

uniformly for z bounded away from 0 and ∞. Therefore, from the definition of $\Gamma_{0,n}$ in Definition 3.1, we know that $\Gamma_{0,n}$ tends to a simple closed curve $\Gamma_{0,\infty}$ encircling 0 once, and

$$\operatorname{Re}(z - 1 - \log z) = 0 \qquad \text{for} \quad z \in \Gamma_{0,\infty}. \tag{3.10}$$

From (1.5), we see that $\Gamma_{0,\infty}$ is the Szegő curve $\mathcal{S}$. □

In [4] a probability measure was defined for each $A < 1$ supported on a set $\Gamma_0 \cup [\beta_1, \beta_2]$, where Γ_0 is a closed contour and $[\beta_1, \beta_2]$ an interval on the positive real line. In the present situation we have that $A = A_n = 1 - \frac{\nu}{n}$ depends on n, but for each finite n we still start from the corresponding probability measure $\mu = \mu_n$ which now depends on n. The measure is given by

$$d\mu_n(y) = \frac{1}{2\pi i}\frac{R_n(y)}{y}\chi_{\Gamma_{0,n}}(y)dy + \frac{1}{2\pi i}\frac{R_{n,+}(y)}{y}\chi_{[\beta_{1,n},\beta_{2,n}]}(y)dy. \tag{3.11}$$

Using (3.6) and (3.8), one can indeed check that (3.11) defines a positive measure on $\Gamma_{0,n} \cup [\beta_{1,n}, \beta_{2,n}]$.

3.2 First transformation $Y \mapsto U$

Expecting the zero distribution as shown in Figure 2 for $0 < A < 1$, we start by modifying the contour in the RH problem. Let

$$\Sigma^U := \Gamma_{0,n} \cup [\beta_{1,n}, \infty). \tag{3.12}$$

We assume (without loss of generality) that the contour Σ in the RH problem for Y was chosen so that $\Gamma_{0,n}$ is contained in Ω_-, see Figures 1 and 4.

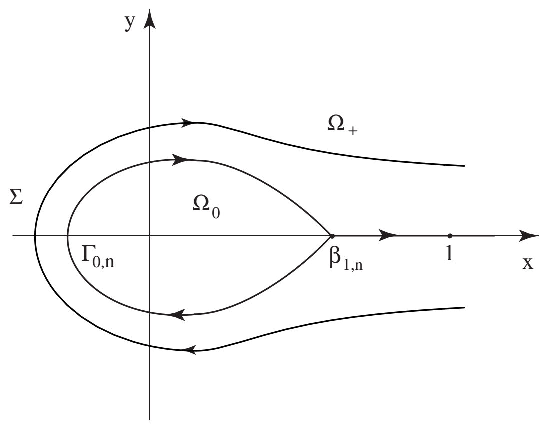

Figure 4: Contours Σ and $\Sigma^U = \Gamma_{0,n} \cup [\beta_{1,n}, \infty)$

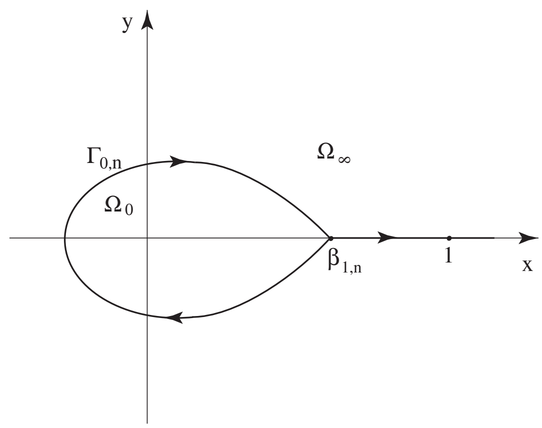

Figure 5: The contour $\Sigma^U = \Gamma_{0,n} \cup [\beta_{1,n}, \infty)$ for the RH problem for U

Introduce U as

$$U(z) = \begin{cases} Y(z) & \text{for } z \in \Omega_+ \cup \Omega_0, \\ Y(z) \begin{pmatrix} 1 & w(z) \\ 0 & 1 \end{pmatrix} & \text{for } z \in \Omega_- \setminus \Omega_0; \end{cases} \tag{3.13}$$

see Figure 4. Then we obtain the following RH problem for U:

(a) $U(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma^U$, see Figure 5;

(b) $U(z)$ possesses continuous boundary values on Σ^U satisfying

$$U_+(z) = U_-(z) \begin{pmatrix} 1 & z^{-n+\nu} e^{-Nz} (z-1)^{2b} \\ 0 & 1 \end{pmatrix} \qquad \text{for } z \in \Gamma_{0,n}, \tag{3.14}$$

$$U_+(x) = U_-(x) \begin{pmatrix} 1 & w_+(x) - w_-(x) \\ 0 & 1 \end{pmatrix} \qquad \text{for } x \in (\beta_{1,n}, 1) \cup (1, \infty); \tag{3.15}$$

(c) $U(z)$ has the following behavior as $z \to \infty$

$$U(z) = \left(I + O\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix};$$

(d) $U(z)$ has the following behavior as $z \to 1$:

$$U(z) \begin{pmatrix} 1 & c_{u,\pm}(z-1)^{2b} \\ 0 & 1 \end{pmatrix} = O(1) \qquad \text{for } z \in \mathbb{C}^{\pm}, \tag{3.16}$$

where

$$c_{u,+} = -e^{-N} \quad \text{and} \quad c_{u,-} = -e^{2\nu\pi i - N}. \tag{3.17}$$

In (3.14) and (3.16), the factor $(z-1)^{2b}$ is defined with a branch cut along $[1, \infty)$ (as before).

Because of the different choices of branches in the definition of $w(z)$ we find that

$$w_+(x) - w_-(x) = |x|^{-n+\nu} e^{-Nx} |x-1|^{2b} (1 - e^{2\nu\pi i}), \qquad x \in (\beta_{1,n}, 1), \tag{3.18}$$

$$w_+(x) - w_-(x) = |x|^{-n+\nu} e^{-Nx} |x-1|^{2b} (1 - e^{2(\nu+2b)\pi i}), \qquad x \in (1, \infty). \tag{3.19}$$

Therefore, (3.15) takes on different forms on the two intervals $(\beta_{1,n}, 1)$ and $(1, \infty)$.

Condition (d) is crucial in the above RH problem for U. With conditions (a)–(c) only, we do not have a unique solution for our RH problem for U. To obtain a unique solution, we have to add the extra condition (d) that controls the local behavior as $z \to 1$. This local behavior (3.16) with the precise constants (3.17) will also be important in the construction of a local parametrix around $z = 1$ that will ultimately be our goal.

3.3 The g and ϕ functions

The measure μ_n in (3.11) gives rise to the so-called g-function, which depends on n,

$$g_n(z) = \int_{\Gamma_{0,n} \cup [\beta_{1,n}, \beta_{2,n}]} \log(z - s) d\mu_n(s), \qquad z \in \mathbb{C} \setminus \Sigma^U, \tag{3.20}$$

where the logarithm $\log(z-s)$ is defined with a branch cut along Σ^U, going to the right from s to ∞. Recalling the definition of $\phi_n(z)$ in (3.6), we similarly define

$$\tilde{\phi}_n(z) = \frac{1}{2}\int_{\beta_{2,n}}^{z} \frac{R_n(s)}{s} ds, \qquad z \in \mathbb{C} \setminus (-\infty, \beta_{2,n}]. \tag{3.21}$$

Note that

$$\phi_n(z) = \tilde{\phi}_n(z) \pm \frac{\nu}{n}\pi i \qquad \text{for } z \in \mathbb{C}^{\pm}; \tag{3.22}$$

see [5, Lemma 3.4.2]. We also introduce

$$g_{t,n}(z) := g_n(z) + (t-1)g_n^0(z) \tag{3.23}$$

and

$$\phi_{t,n}(z) := \phi_n(z) + (t-1)\phi_n^0(z), \qquad \tilde{\phi}_{t,n}(z) := \tilde{\phi}_n(z) + (t-1)\phi_n^0(z), \tag{3.24}$$

where

$$g_n^0(z) = \begin{cases} \frac{1}{2}(z - R_n(z)), & z \in \Omega_\infty, \\ \frac{1}{2}(z + R_n(z)), & z \in \Omega_0, \end{cases} \tag{3.25}$$

and

$$\phi_n^0(z) = \frac{R_n(z)}{2}, \qquad z \in \mathbb{C} \setminus [\beta_{1,n}, \beta_{2,n}]. \tag{3.26}$$

Note that $g_n^0(z) = O(1/z)$ as $z \to \infty$. It is easily seen that the functions $g_n^0(z)$ and $\phi_n^0(z)$ satisfy the following properties.

Proposition 3.3. (a) *We have*

$$g_n^0(z) = \frac{1}{2}\,z + \begin{cases} -\phi_n^0(z), & z \in \Omega_\infty \cap \mathbb{C}^{\pm}, \\ +\phi_n^0(z), & z \in \Omega_0 \cap \mathbb{C}^{\pm}. \end{cases} \tag{3.27}$$

(b) *For $z \in \Sigma^U$, we have*

$$g_{n,+}^0(z) - g_{n,-}^0(z) = \begin{cases} -2\phi_n^0(z), & z \in \Gamma_{0,n} \cap \mathbb{C}_{\pm}, \\ -2\phi_{n,+}^0(z) = 2\phi_{n,-}^0(z), & z \in (\beta_{1,n}, \beta_{2,n}]. \end{cases} \tag{3.28}$$

(c) *We also have*

$$g_{n,+}^0(z) + g_{n,-}^0(z) = \begin{cases} z, & z \in \Gamma_{0,n} \cup [\beta_{1,n}, \beta_{2,n}], \\ z - 2\phi_n^0(z), & z \in [\beta_{2,n}, \infty). \end{cases} \tag{3.29}$$

Then from the above proposition and Proposition 3.2 in [4], we get the following properties for $g_{t,n}$, $\phi_{t,n}$, and $\tilde{\phi}_{t,n}$.

Proposition 3.4. (a) *There exists a constant* $l_{t,n}$ *such that*

$$g_{t,n}(z) = \frac{1}{2}\left(A_n \log z + tz + l_{t,n}\right) + \begin{cases} \mp\frac{1}{2}A_n\pi i - \phi_{t,n}(z), & z \in \Omega_\infty \cap \mathbb{C}^\pm, \\ \pm\frac{1}{2}A_n\pi i + \phi_{t,n}(z), & z \in \Omega_0 \cap \mathbb{C}^\pm. \end{cases} \tag{3.30}$$

Here $\log z$ *is defined with a cut along* $[0, \infty)$*. The constant* $l_{t,n}$ *is explicitly given by*

$$l_{t,n} = 2g_{t,n}(x_n) - (A_n \log x_n + x_n),$$

where x_n *is the intersection of* $\Gamma_{0,n}$ *with the negative real axis.*

(b) *For* $z \in \Sigma^U$*, we have*

$$g_{t,n,+}(z) - g_{t,n,-}(z) = \begin{cases} -\phi_{t,n}(z) - \tilde{\phi}_{t,n}(z) \mp \pi i, & z \in \Gamma_{0,n} \cap \mathbb{C}_\pm, \\ -2\phi_{t,n,+}(z) - 2A_n\pi i = 2\phi_{t,n,-}(z) - 2A_n\pi i, & z \in (\beta_{1,n}, \beta_{2,n}], \\ -2\pi i, & z \in [\beta_{2,n}, \infty). \end{cases} \tag{3.31}$$

(c) *We also have*

$$g_{t,n,+}(z) + g_{t,n,-}(z) = \begin{cases} A_n \log z + tz + l_{t,n}, & z \in \Gamma_{0,n}, \\ A_n \log z + tz + A_n\pi i + l_{t,n}, & z \in [\beta_{1,n}, \beta_{2,n}], \\ A_n \log z + tz + A_n\pi i + l_{t,n} - 2\tilde{\phi}_{t,n}(z), & z \in [\beta_{2,n}, \infty). \end{cases} \tag{3.32}$$

3.4 Second transformation $U \mapsto T$

To normalize the RH problem at infinity, introduce the second transformation $U \mapsto T$ as

$$T(z) = \Big((-1)^n(e^{-i\pi\nu} - e^{i\pi\nu})\Big)^{-\frac{1}{2}\sigma_3} e^{-\frac{1}{2}nl_{t,n}\sigma_3} U(z) \\ \times e^{-ng_{t,n}(z)\sigma_3} e^{\frac{1}{2}nl_{t,n}\sigma_3} \Big((-1)^n(e^{-i\pi\nu} - e^{i\pi\nu})\Big)^{\frac{1}{2}\sigma_3}. \tag{3.33}$$

Note that it is important here that ν is not an integer, since this assumption guarantees that $e^{-i\pi\nu} - e^{i\pi\nu} \neq 0$ and therefore T is well-defined.

With the RH problem for U and Propositions 3.3 and 3.4, direct calculation gives us a RH problem for T as follows. As before $(z-1)^{2b}$ is defined with a cut along $[1,\infty)$.

(a) $T(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma^U$;

(b)

$$T_+(z) = T_-(z) \begin{pmatrix} (-1)^n e^{n(\phi_{t,n}(z)+\tilde{\phi}_{t,n}(z))} & (-1)^n (e^{-\nu\pi i} - e^{\nu\pi i})^{-1} (z-1)^{2b} \\ 0 & (-1)^n e^{-n(\phi_{t,n}(z)+\tilde{\phi}_{t,n}(z))} \end{pmatrix}, z \in \Gamma_{0,n}; \tag{3.34}$$

$$T_+(z) = T_-(z) \begin{pmatrix} e^{2n\tilde{\phi}_{t,n,+}(z)} & (z-1)^{2b} \\ 0 & e^{2n\tilde{\phi}_{t,n,-}(z)} \end{pmatrix}, \qquad z \in (\beta_{1,n}, 1); \tag{3.35}$$

$$T_+(z) = T_-(z) \begin{pmatrix} e^{2n\tilde{\phi}_{t,n,+}(z)} & |z-1|^{2b} \frac{\sin(\nu+2b)\pi}{\sin \nu\pi} e^{2b\pi i} \\ 0 & e^{2n\tilde{\phi}_{t,n,-}(z)} \end{pmatrix}, \qquad z \in (1, \beta_{2,n}); \tag{3.36}$$

$$T_+(z) = T_-(z) \begin{pmatrix} 1 & e^{-2n\tilde{\phi}_{t,n}(z)} |z-1|^{2b} \frac{\sin(\nu+2b)\pi}{\sin \nu\pi} e^{2b\pi i} \\ 0 & 1 \end{pmatrix}, \qquad z \in (\beta_{2,n}, \infty); \tag{3.37}$$

(c) $T(z)$ has the following behavior as $z \to \infty$

$$T(z) = I + O\left(\frac{1}{z}\right);$$

(d) $T(z)$ has the following behavior as $z \to 1$:

$$T(z) \begin{pmatrix} 1 & c_{t,\pm}(z-1)^{2b} \\ 0 & 1 \end{pmatrix} = O(1) \qquad \text{for } z \in \mathbb{C}^{\pm} \tag{3.38}$$

with

$$c_{t,\pm} = (-1)^n (e^{-\nu\pi i} - e^{\nu\pi i})^{-1} \, e^{2n g_{t,n,\pm}(1) - n l_{t,n}} c_{u,\pm}, \tag{3.39}$$

where $c_{u,\pm}$ is given in (3.17).

To derive the jump matrices for T in (3.35)–(3.37), one needs to make use of (3.15), (3.18), (3.19) and Proposition 3.4. The conjugation with the constant matrix $\left((-1)^n (e^{-i\pi\nu} - e^{i\pi\nu})\right)^{-\frac{1}{2}\sigma_3}$ in (3.33) may look a bit awkward since it makes the jump matrix in (3.34) more complicated. We have introduced it in order to simplify the jump matrices (3.35)–(3.37). We also want to point out that the entry $\frac{\sin(\nu+2b)\pi}{\sin \nu\pi}$ in

(3.36) and (3.37) will come out in the Stokes multipliers s_j's in the RH problem for PIV later on.

By using the relations between the g-functions and ϕ-functions in (3.30) and (3.27), the constant $c_{t,\pm}$ in (3.39) can be rewritten as

$$c_{t,\pm} = -(e^{-\nu\pi i} - e^{\nu\pi i})^{-1}\, e^{-2n\phi_{t,n,\pm}(1)\pm\nu\pi i}. \tag{3.40}$$

3.5 Third transformation $T \mapsto S$

From (3.8), (3.22) and (3.24), we know that, when $|t-1|$ is small, the real part of $\phi_{t,n}(z)+\tilde{\phi}_{t,n}(z)$ is close to 0 for $z \in \Gamma_{0,n}$. This means the jump matrix for T on $\Gamma_{0,n}$ in (3.34) has oscillatory diagonal entries. To remove this oscillating jump, we introduce the third transformation and open the lens as follows. In a fixed neighborhood of $z = 1$ let the curve Γ_1 be defined as $\operatorname{Im} z = \arg z$, then we continue Γ_1 such that it becomes a closed contour with the Szegő curve $\mathcal{S}$ in its interior. Note that Γ_1 is independent of n and t. Then Γ_1 and $\Gamma_{0,n}$ divide the complex plane into three domains Ω_0, Ω_1 and Ω_∞^S; see Figure 6. Define

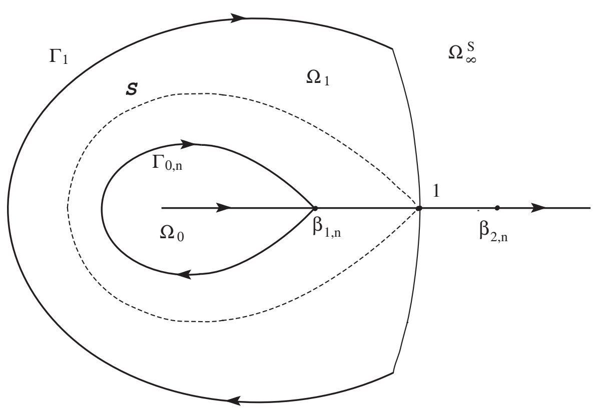

Figure 6: Szegő curve and the contour for the RH problem for S

$$S(z) = T(z)\begin{pmatrix} 0 & (-1)^n \frac{(z-1)^{2b}}{e^{-\nu\pi i}-e^{\nu\pi i}} \\ (-1)^{n+1}\frac{e^{-\nu\pi i}-e^{\nu\pi i}}{(z-1)^{2b}} & (-1)^n e^{-n(\phi_{t,n}(z)+\tilde{\phi}_{t,n}(z))} \end{pmatrix}, \quad z \in \Omega_0; \tag{3.41}$$

$$S(z) = T(z)\begin{pmatrix} 1 & 0 \\ -\frac{e^{-\nu\pi i}-e^{\nu\pi i}}{(z-1)^{2b}}\, e^{n(\phi_{t,n}(z)+\tilde{\phi}_{t,n}(z))} & 1 \end{pmatrix}, \quad z \in \Omega_1; \tag{3.42}$$

$$S(z) = T(z), \quad z \in \Omega_\infty^S. \tag{3.43}$$

Direct calculation shows that $S_+(z) = S_-(z)$ for $z \in \Gamma_{0,n}$. Therefore, $S(z)$ has an analytic continuation across $\Gamma_{0,n}$ and we obtain a RH problem for S on the contour

$\Sigma^S = \Gamma_1 \cup [0,\infty)$ shown in Figure 7. We use Ω_0^S to denote the bounded domain enclosed by Γ_1.

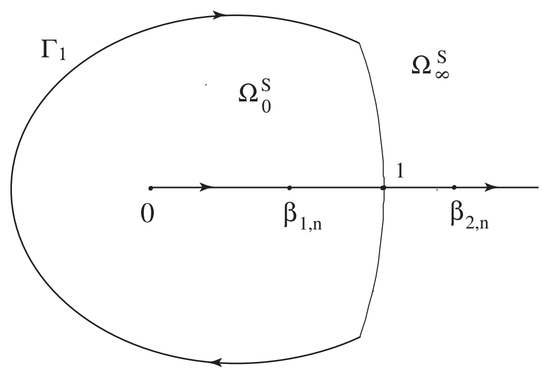

Figure 7: The contour Σ^S for the RH problem for S

(a) S is analytic for $z \in \mathbb{C} \setminus \Sigma^S$, see Figure 7;

(b)

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & 0 \\ \frac{e^{-\nu\pi i} - e^{\nu\pi i}}{(z-1)^{2b}} e^{n(\phi_{t,n}(z)+\tilde{\phi}_{t,n}(z))} & 1 \end{pmatrix}, \qquad z \in \Gamma_1; \quad (3.44)$$

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & (z-1)^{2b} e^{-2n\phi_{t,n}(z)} \\ 0 & 1 \end{pmatrix}, \qquad z \in (0, \beta_{1,n}); \quad (3.45)$$

$$S_+(z) = S_-(z) \begin{pmatrix} e^{2n\phi_{t,n,+}(z)} & (z-1)^{2b} \\ 0 & e^{2n\phi_{t,n,-}(z)} \end{pmatrix}, \qquad z \in (\beta_{1,n}, 1); \quad (3.46)$$

$$S_+(z) = S_-(z) \begin{pmatrix} e^{2n\tilde{\phi}_{t,n,+}(z)} & |z-1|^{2b} \frac{\sin(\nu+2b)\pi}{\sin\nu\pi} e^{2b\pi i} \\ 0 & e^{2n\tilde{\phi}_{t,n,-}(z)} \end{pmatrix}, \qquad z \in (1, \beta_{2,n}); \quad (3.47)$$

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & |z-1|^{2b} \frac{\sin(\nu+2b)\pi}{\sin\nu\pi} e^{2b\pi i} e^{-2n\tilde{\phi}_{t,n}(z)} \\ 0 & 1 \end{pmatrix}, \quad z \in (\beta_{2,n}, \infty); \quad (3.48)$$

(c) $S(z)$ has the following behavior as $z \to \infty$

$$S(z) = I + O\left(\frac{1}{z}\right);$$

(d) $S(z)$ has the following behavior as $z \to 1$

$$S(z) \begin{pmatrix} 1 & c_{t,\pm}(z-1)^{2b} \\ -c_{t,\pm}^{-1}(z-1)^{-2b} & 0 \end{pmatrix} = O(1), \; z \in \Omega_0^S \cap \mathbb{C}^\pm, \quad (3.49)$$

$$S(z) \begin{pmatrix} 1 & c_{t,\pm}(z-1)^{2b} \\ 0 & 1 \end{pmatrix} = O(1), \; z \in \Omega_\infty^S \cap \mathbb{C}^\pm, \quad (3.50)$$

where $c_{t,\pm}$ is given in (3.40).

The jump matrices in (3.44)–(3.48) look quite complicated. However, most of them tend to the identity matrix as $n \to \infty$ at an exponential rate. To clarify this fact, we need the following proposition.

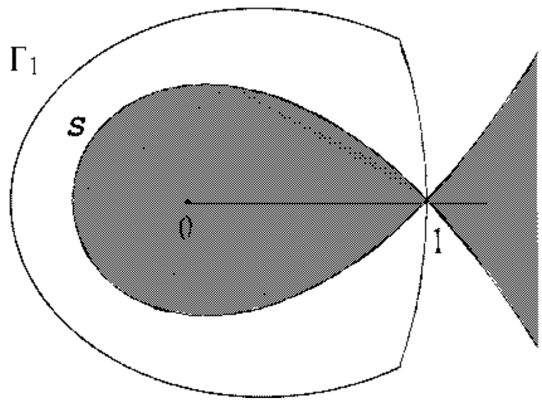

Figure 8: $\mathcal{S}$ is the Szegő curve. The dark region is the region where $\mathrm{Re}(z-1-\log z) > 0$, the white region is the region where $\mathrm{Re}(z-1-\log z) < 0$.

Proposition 3.5. *Let δ be a fixed small constant and $B_\delta := \{z \in \mathbb{C} \mid |z-1| < \delta\}$. Then there exist $\eta > 0$ and $\varepsilon > 0$ such that for all $t \in \mathbb{R}$ with $|t-1| < \eta$ and all n large enough, we have*

$$\mathrm{Re}\, \phi_{t,n}(z) < -\varepsilon \qquad \text{for } z \in \Gamma_1 \setminus B_\delta$$

and

$$\mathrm{Re}\, \phi_{t,n}(z) > \varepsilon(|z|+1) \qquad \text{for } z \in (0,\infty) \setminus B_\delta.$$

The same inequalities hold for $\mathrm{Re}\, \tilde{\phi}_{t,n}(z)$.

Proof. First recall from (3.9), that $\phi_n(z) \to \frac{1}{2}(z-1-\log z)$ uniformly for z bounded away from 0 and ∞; see Figure 8 for the property of $\mathrm{Re}(z-1-\log z)$. For z near 0 and ∞ we have

$$\phi_n(z) = \begin{cases} -\log z + O(1) & \text{as } z \to 0, \\ z + O(1) & \text{as } z \to \infty, \end{cases}$$

both uniformly for n large enough. Since both the contour Γ_1 and the constant δ are independent of n, there exists an $\varepsilon > 0$ such that $\mathrm{Re}\, \phi_n(z) < -\varepsilon$ on $\Gamma_1 \setminus B_\delta$ and $\mathrm{Re}\, \phi_n(z) > \varepsilon(|z|+1)$ on $(0,\infty) \setminus B_\delta$ uniformly for n large enough. From the fact that

$$\phi_n^0(z) = \frac{R_n(z)}{2} \to \frac{z-1}{2} \qquad \text{as } n \to \infty, \quad \text{unifomly for all } z,$$

and the definition of $\phi_{t,n}(z)$ in (3.24), it follows by continuity that the same inequalities hold for $\phi_{t,n}(z)$ if t is sufficiently close to 1.

Since $\operatorname{Re}\phi_n(z) = \operatorname{Re}\tilde{\phi}_n(z)$ for all $z \in \mathbb{C}$, the inequalities also hold if we replace $\phi_n(z)$ by $\tilde{\phi}_n(z)$. $\square$

3.6 Construction of the local parametrix

3.6.1 RH problem for P

From Proposition 3.5, we know that the jump matrices for S are exponentially close to the identity matrix if n is large, except for the ones in the neighborhood of $z = 1$. Now, let us focus on the RH problem of S restricted to a neighborhood B_δ of $z = 1$. We seek a 2×2 matrix valued function P that satisfies the following RH problem.

(a) $P(z)$ is analytic for $z \in B_\delta \setminus \Sigma^S$, and continuous on $\overline{B}_\delta \setminus \Sigma^S$;

(b) $P(z)$ satisfies the same jump conditions on $\Sigma^S \cap B_\delta$ as S does; see (3.44)–(3.48);

(c) on ∂B_δ, as $n \to \infty$

$$P(z) = \left(I + O\left(\frac{1}{\sqrt{n}}\right)\right) n^{\frac{b}{2}\sigma_3} \qquad \text{uniformly for } z \in \partial B_\delta \setminus \Sigma^S; \tag{3.51}$$

(d) $P(z)$ satisfies the same local behavior near $z = 1$ as S does; see (3.49) and (3.50) with the same constants $c_{t,\pm}$ as given in (3.40).

The factor $n^{\frac{b}{2}\sigma_3}$ in the matching condition (3.51) is special and unusual in the local parametrix construction. It comes out of our construction with the RH problem for Ψ from Section 1.2 and we are not able to remove or simplify it. However, we can deal with the factor in the final transformation in Section 3.7 below.

3.6.2 Reduction to constant jumps

We look for P in the following form

$$P(z) = \begin{cases} \sigma_1 Q(z) \sigma_1 e^{(n\phi_{t,n}(z) - \frac{1}{2}(\nu+b)\pi i)\sigma_3} (z-1)^{-b\sigma_3} & \text{for } z \in B_\delta \cap \Omega_0^S, \\ \sigma_1 Q(z) \sigma_1 e^{(n\tilde{\phi}_{t,n}(z) + \frac{1}{2}(\nu+b)\pi i)\sigma_3} (1-z)^{-b\sigma_3} & \text{for } z \in B_\delta \cap \Omega_\infty^S, \end{cases} \tag{3.52}$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $(z-1)^b$ and $(1-z)^b$ are defined with branch cuts along $[1, \infty)$ and $(-\infty, 1]$, respectively. Thus we have $0 < \arg(z-1) < 2\pi$, $0 < \arg(1-z) < 2\pi$ and

$$\arg(z-1) = \arg(1-z) \mp \pi i \qquad \text{for } z \in \mathbb{C}^\pm. \tag{3.53}$$

The factors $e^{-n\phi_{t,n}(z)\sigma_3}(z-1)^{b\sigma_3}$ and $e^{-n\tilde{\phi}_{t,n}(z)\sigma_3}(1-z)^{b\sigma_3}$ in (3.52) are introduced to cancel the corresponding factors in the jump matrices in (3.44)–(3.48) and make them independent of z. The result is that Q will satisfy a RH problem with constant jumps.

Reorient the contour $\Sigma^S \cap B_\delta$ so that all parts are oriented away from the point $z = 1$; see Figure 9. Then it can be verified that, in order for $P(z)$ to satisfy the desired RH problem, $Q(z)$ should satisfy a RH problem as follows:

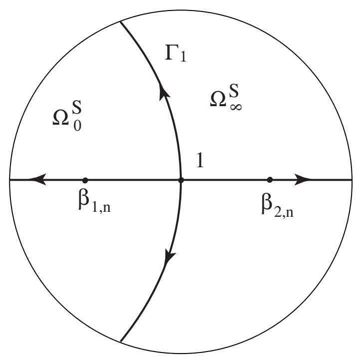

Figure 9: The contour $\Sigma^S \cap B_\delta$ for the RH problem for Q

(a) $Q(z)$ is analytic for $z \in B_\delta \setminus \Sigma^S$, and continuous on $\overline{B}_\delta \setminus \Sigma^S$;

(b)

$$Q_+(z) = Q_-(z)\begin{pmatrix} 1 & 0 \\ \frac{\sin(\nu+2b)\pi}{\sin \nu\pi} e^{(\nu+b)\pi i} & 1 \end{pmatrix}, \qquad z \in (1, 1+\delta); \quad (3.54)$$

$$Q_+(z) = Q_-(z)\begin{pmatrix} 1 & (e^{\nu\pi i} - e^{-\nu\pi i})e^{b\pi i} \\ 0 & 1 \end{pmatrix}, \qquad z \in \Gamma_1 \cap B_\delta \cap \mathbb{C}^+; \quad (3.55)$$

$$Q_+(z) = Q_-(z)\begin{pmatrix} 1 & 0 \\ -e^{-(\nu+b)\pi i} & 1 \end{pmatrix}, \qquad z \in (1-\delta, 1); \quad (3.56)$$

$$Q_+(z) = Q_-(z)\begin{pmatrix} 1 & (e^{-\nu\pi i} - e^{\nu\pi i})e^{(2\nu+b)\pi i} \\ 0 & 1 \end{pmatrix} e^{2(\nu+b)\pi i\sigma_3}, \quad z \in \Gamma_1 \cap B_\delta \cap \mathbb{C}^-; \quad (3.57)$$

(c) for $z \in \partial B_\delta$ as $n \to \infty$,

$$Q(z) = \left(I + O\left(\frac{1}{\sqrt{n}}\right)\right)(\sqrt{n}(z-1))^{-(\nu+b)\sigma_3} \exp\left[\frac{n}{2}(z - 1 - \log z) + \frac{n^{\frac{1}{2}} L}{\sqrt{2}}(z-1) + \frac{\nu}{2}(\log z + \log \nu - 1) + \frac{1}{2}(\nu - b)\pi i\right]\sigma_3 \quad (3.58)$$

where $\log z$ and $(z-1)^{-(\nu+b)}$ are defined with cuts along $(-\infty,0]$ and along $\Gamma_1\cap B_\delta\cap\mathbb{C}^-$, respectively;

(d) $Q(z)$ has the following behavior as $z\to 1$:

$$Q(z)\begin{pmatrix} 0 & (e^{-\nu\pi i}-e^{\nu\pi i})e^{b\pi i} \\ \frac{e^{-b\pi i}}{e^{\nu\pi i}-e^{-\nu\pi i}} & 1 \end{pmatrix}(z-1)^{b\sigma_3}=O(1),\quad z\in\Omega_0^S\cap\mathbb{C}^+; \tag{3.59}$$

$$Q(z)\begin{pmatrix} 0 & (e^{-\nu\pi i}-e^{\nu\pi i})e^{b\pi i} \\ \frac{e^{-b\pi i}}{e^{\nu\pi i}-e^{-\nu\pi i}} & e^{-2\nu\pi i} \end{pmatrix}(z-1)^{b\sigma_3}=O(1),\quad z\in\Omega_0^S\cap\mathbb{C}^-; \tag{3.60}$$

$$Q(z)\begin{pmatrix} 1 & 0 \\ \frac{e^{-b\pi i}}{e^{\nu\pi i}-e^{-\nu\pi i}} & 1 \end{pmatrix}(z-1)^{b\sigma_3}=O(1),\quad z\in\Omega_\infty^S\cap\mathbb{C}^+; \tag{3.61}$$

$$Q(z)\begin{pmatrix} 1 & 0 \\ \frac{e^{(2\nu+3b)\pi i}}{e^{\nu\pi i}-e^{-\nu\pi i}} & 1 \end{pmatrix}(z-1)^{b\sigma_3}=O(1),\quad z\in\Omega_\infty^S\cap\mathbb{C}^-; \tag{3.62}$$

where now $(z-1)^b$ is defined with a cut along $\Gamma_1\cap B_\delta\cap\mathbb{C}^-$.

In the above RH problem for Q, it is notable that, although $\beta_{2,n}\in(1,1+\delta)$, the jump matrix in (3.54) is the same for $z<\beta_{2,n}$ and $z>\beta_{2,n}$. This is in contrast to the RH problem for P, where we have different expressions for $z<\beta_{2,n}$ and $z>\beta_{2,n}$. A similar thing happens for the jump matrix in (3.56).

To obtain the matching condition in (3.58), we need to make use of the following asymptotic formulas for $\phi_n^0(z)$ and $\phi_n(z)$ as $n\to\infty$. For $z\in\partial B_\delta$, we have from (3.26)

$$\phi_n^0(z)=\frac{1}{2}(z-1)-\frac{\nu(z+1)}{2n(z-1)}+O\left(\frac{1}{n^2}\right) \tag{3.63}$$

and from (3.6), calculated with the assistance of Maple,

$$\begin{aligned}\phi_n(z)=&\frac{1}{2}(z-1-\log z)-\frac{\nu}{2n}\log n+\frac{\nu}{2n}\Big(-2\log(z-1)+2\pi i+\log z+\log\nu-1\Big)\\&+\frac{\nu^2}{n^2(z-1)^2}+O\left(\frac{1}{n^3}\right),\end{aligned} \tag{3.64}$$

where $\log z$ and $\log(z-1)$ are defined with cuts along $(-\infty,0]$ and $[1,\infty)$, respectively; see also [5] and [6]. For $z\in\partial B_\delta\cap\Omega_0^S$, we have from (3.24), (3.51) and (3.52)

$$Q(z)=\left(I+O(\frac{1}{\sqrt{n}})\right)(\sqrt{n}(z-1))^{-b\sigma_3}e^{(n(\phi_n(z)+(t-1)\phi_n^0(z))-\frac{1}{2}(\nu+b)\pi i)\sigma_3}.$$

Substituting (3.63) and (3.64) into the above formula, we get

$$Q(z) = \left(I + O(\frac{1}{\sqrt{n}})\right)(\sqrt{n}(z-1))^{-(\nu+b)\sigma_3} \exp\bigg[\frac{n}{2}(z - 1 - \log z) \\ + \frac{\nu}{2}(2\pi i + \log z + \log \nu - 1) + O(\frac{1}{n}) + \frac{n(t-1)}{2}\left((z-1) + O(\frac{1}{n})\right) - \frac{1}{2}(\nu+b)\pi i\bigg]\sigma_3,$$

where $\log z$ and $(z-1)^{-(\nu+b)}$ are defined with cuts along $(-\infty, 0]$ and $\Gamma_1 \cap B_\delta \cap \mathbb{C}^-$, respectively. Recalling the definition of t in (3.1), the above formula immediately gives us (3.58) for $z \in \partial B_\delta \cap \Omega_0^S$. Similarly, we can prove that (3.58) also holds for $z \in \partial B_\delta \cap \Omega_\infty^S$.

To obtain the limiting behaviors (3.59)–(3.62) near $z = 1$, some careful calculations are needed. We take (3.62) as an example and (3.59)–(3.61) can be obtained in a similar way. From (3.52), (3.53) and condition (d) for the RH problem for P, we have, as $z \to 1, z \in \Omega_\infty^S \cap \mathbb{C}^-$

$$Q(z)e^{-(n\tilde{\phi}_{t,n,-}(1)+\frac{1}{2}(\nu+b)\pi i)\sigma_3}((z-1)e^{-\pi i})^{b\sigma_3}\begin{pmatrix} 1 & 0 \\ c_{t,-}(z-1)^{2b} & 1 \end{pmatrix} = O(1), \tag{3.65}$$

where $(z-1)^b$ is defined with a cut along $[1, \infty)$ and $c_{t,-}$ is given in (3.40). Changing the branch cut of $(z-1)^b$ to $\Gamma_1 \cap B_\delta \cap \mathbb{C}^-$, we get

$$(z-1) \mapsto (z-1)e^{2\pi i} \qquad \text{for } z \in \Omega_\infty^S \cap \mathbb{C}^-. \tag{3.66}$$

From (3.65) and (3.66), it follows that, as $z \to 1, z \in \Omega_\infty^S \cap \mathbb{C}^-$

$$Q(z)e^{-(n\tilde{\phi}_{t,n,-}(1)+\frac{1}{2}(\nu+b)\pi i)\sigma_3}((z-1)e^{\pi i})^{b\sigma_3}\begin{pmatrix} 1 & 0 \\ c_{t,-}(z-1)^{2b}e^{4b\pi i} & 1 \end{pmatrix} = O(1).$$

Right multiplying $e^{(n\tilde{\phi}_{t,n,-}(1)+\frac{1}{2}(\nu+b)\pi i)\sigma_3}e^{-b\pi i\sigma_3}$ on both sides of the above equation yields, as $z \to 1, z \in \Omega_\infty^S \cap \mathbb{C}^-$

$$Q(z)(z-1)^{b\sigma_3}\begin{pmatrix} 1 & 0 \\ c_{t,-}(z-1)^{2b}e^{2n\tilde{\phi}_{t,n,-}(1)+(\nu+3b)\pi i} & 1 \end{pmatrix} = O(1).$$

Recalling (3.22), (3.40) and moving $(z-1)^{b\sigma_3}$ to the right of the lower triangular matrix, we obtain (3.62).

3.6.3 Comparison with the RH problem for Ψ

Now we compare the RH problem for Q with the RH problem for Ψ in Section 1.2. From (3.54)–(3.57) we see that we need the Stokes multipliers

$$\begin{aligned} s_1 &= \frac{\sin(\nu+2b)\pi}{\sin\nu\pi} e^{(\nu+b)\pi i}, \quad & s_2 &= (e^{\nu\pi i} - e^{-\nu\pi i}) e^{b\pi i}, \\ s_3 &= -e^{-(\nu+b)\pi i}, & s_4 &= (e^{-\nu\pi i} - e^{\nu\pi i}) e^{(2\nu+b)\pi i}. \end{aligned} \tag{3.67}$$

Because of the factors $(z-1)^{b\sigma_3}$ in (3.59)–(3.62) and $(z-1)^{-(\nu+b)\sigma_3}$ in (3.58), we are led to choose

$$\Theta = -b, \qquad \Theta_\infty = \nu + b. \tag{3.68}$$

Then the above Stokes multipliers are obtained from those in (1.28), by applying the transformation in (1.27) with

$$d = (e^{\nu\pi i} - e^{-\nu\pi i})^{-1}.$$

Again it is important that ν is not an integer, since this ensures $d \neq \infty$. Furthermore, from (1.31), we get

$$\mathcal{C}^{-1} = \begin{pmatrix} 1 & 0 \\ de^{(2\Theta_\infty - \Theta)\pi i} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{e^{(2\nu+3b)\pi i}}{e^{\nu\pi i} - e^{-\nu\pi i}} & 1 \end{pmatrix},$$

which appears exactly in (3.62). Thus the jump matrices and the local behavior near the point 1 in the RH problem for Q correspond exactly to the jump matrices and the behavior near the origin in the RH problem for Ψ.

From (3.59)–(3.62), we do not want a logarithmic singularity at point 1 for Q for all values of b. This is special when we want to construct our parametrix for Q by using the Ψ-function in Section 1.2. Note that $l = 0$ is a Fuchsian singular point of the equation $\frac{\partial \Psi}{\partial l} = A\Psi$ in (1.11) and the leading term of A as $l \to 0$ is

$$\frac{1}{l}\left[(\Theta - K)\sigma_3 - \frac{u\,y}{2}\sigma_+ + \frac{2K}{u\,y}(K - 2\Theta)\sigma_- \right].$$

The eigenvalues of the above coefficient matrix are Θ and $-\Theta$. When $\Theta = m/2, m \in \mathbb{Z}$, usually this is a resonant case and there is a logarithmic singularity at 0 for Ψ; see [28, Section 1.3]. In this paper, we are going to make use of Ψ-function which is free from logarithmic singularities. Then with $\Theta = -b$, $\Theta_\infty = \nu + b$ and the Stokes multipliers given in (3.67), in Section 6 we will see that there exist special function solutions to PIV when $b = m/2, m \in \mathbb{Z}$. Also see [32] for similar cases for Painlevé II solutions.

3.6.4 Construction of Q

Using the RH problem for Ψ, we now construct Q as follows. We use the mapping function

$$f(z) = [z - 1 - \log z]^{1/2} \tag{3.69}$$

which is a conformal map from a neighborhood of 1 onto a neighborhood of 0. We have

$$f(z) = \frac{1}{\sqrt{2}}(z-1) - \frac{\sqrt{2}}{6}(z-1)^2 + O((z-1)^3) \qquad \text{as } z \to 1. \tag{3.70}$$

Since for $z \in \Gamma_1 \cap B_\delta$ we have that $\operatorname{Im} z = \arg z$, it can be shown that f maps $\Gamma_1 \cap B_\delta$ to the imaginary axis.

Now we define

$$Q(z) = E(z)\Psi\left(n^{\frac{1}{2}} f(z), L\frac{z-1}{\sqrt{2}f(z)}\right) \tag{3.71}$$

where

$$E(z) = \left(\frac{f(z)}{z-1}\right)^{(\nu+b)\sigma_3} (\nu e^{-1}\ z)^{\frac{\nu}{2}\sigma_3} e^{\frac{1}{2}(\nu-b)\pi i\sigma_3}, \tag{3.72}$$

L is given in (1.33) and $\Psi(\cdot,\cdot)$ is the solution of the RH problem (1.18)–(1.24) with Θ, Θ_∞ given in (3.68) and Stokes multipliers s_j given in (3.67). Because of (3.70) we have that E is analytic in a (small enough) neighborhood of $z = 1$. We also see that

$$L\frac{z-1}{\sqrt{2}f(z)} \to L \qquad \text{as } z \to 1,$$

Since L is (by assumption) not a pole of the PIV solution $u(s)$, there is a small enough $\delta > 0$ (depending on L) so that

$$z \mapsto s = L\frac{z-1}{\sqrt{2}f(z)}$$

maps $\overline{B}_\delta$ to a compact subset of the s-plane that does not contain any poles of $u(s)$. Then $\Psi(\lambda, L\frac{z-1}{\sqrt{2}f(z)})$ exists for all $z \in B_\delta$, and $Q(z)$ is well-defined and analytic for $z \in B_\delta \setminus \Sigma^S$.

It is now easy to check that Q satisfies the jump conditions (3.54)–(3.57). We also find that the behavior as $z \to 1$ in (3.62) is satisfied. This follows from the corresponding behavior as $\lambda \to 0$ in the RH problem for Ψ. The behaviors as $z \to 1$ in (3.59)–(3.61) follow from this and the jump conditions in the RH problem for Ψ.

So what remains is to check the matching condition (3.58) in the RH problem for Q. For this we need to note that the asymptotic condition (1.23) holds uniformly for

s in compact sets away from the poles of $u(s)$. Thus by the definition (3.71) we have for $z \in \partial B_\delta$ as $n \to \infty$

$$Q(z) = E(z)\left(I + O\left(\frac{1}{\sqrt{n}}\right)\right)\exp\left[\frac{n}{2}f^2(z)\,\sigma_3 + \frac{n^{\frac{1}{2}}\sqrt{2}\,L(z-1)}{2}\,\sigma_3\right]\left(\frac{1}{n^{1/2}f(z)}\right)^{(\nu+b)\sigma_3},$$

where $f(z)^{-(\nu+b)}$ is defined with a cut along $\Gamma_1 \cap B_\delta \cap \mathbb{C}^-$. Since $E(z)$ is independent of n, then

$$Q(z) = \left(I + O\left(\frac{1}{\sqrt{n}}\right)\right)\exp\left[\frac{n}{2}f^2(z)\,\sigma_3 + \frac{n^{\frac{1}{2}}\sqrt{2}\,L(z-1)}{2}\,\sigma_3\right]\left(\frac{1}{n^{1/2}f(z)}\right)^{(\nu+b)\sigma_3} E(z).$$

From the definition of $E(z)$ in (3.72), we get

$$\begin{aligned} Q(z) = &\left(I + O\left(\frac{1}{\sqrt{n}}\right)\right)(\sqrt{n}(z-1))^{-(\nu+b)\sigma_3}\exp\left[\frac{n}{2}f^2(z) + \frac{n^{\frac{1}{2}}\sqrt{2}\,L(z-1)}{2}\right. \\ &\left. + \frac{\nu}{2}(\log z + \log\nu - 1) + \frac{1}{2}(\nu - b)\pi i\right]\sigma_3, \end{aligned} \tag{3.73}$$

where $\log z$ and $(z-1)^{-(\nu+b)}$ are defined with cuts along $(-\infty, 0]$ and $\Gamma_1 \cap B_\delta \cap \mathbb{C}^-$, respectively. Recalling the definition of $f(z)$ in (3.69), we obtain (3.58) from (3.73).

This completes the construction of Q.

3.7 Final transformation

Now let us consider the difference between the exact solution S and the parametrix P constructed. Define

$$R(z) = \begin{cases} n^{\frac{b}{2}\sigma_3}S(z)n^{-\frac{b}{2}\sigma_3}, & z \in \mathbb{C} \setminus (\overline{B}_\delta \cup \Sigma^S), \\ n^{\frac{b}{2}\sigma_3}S(z)P^{-1}(z), & z \in B_\delta \setminus \Sigma^S. \end{cases} \tag{3.74}$$

From the RH problems for S and P, since P is analytic for $z \in B_\delta \setminus \Sigma^S$ and satisfies the same jump conditions on $\Sigma^S \cap B_\delta$ as S does, $R(z)$ is analytic for $z \in B_\delta$ except for a possible pole at $z = 1$. Moreover, because S and P have the same local behavior near $z = 1$, $R(z)$ has to be analytic at $z = 1$. Thus, we get a RH problem for $R(z)$ on a contour Σ^R as follows:

(a) $R(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma^R$; see Figure 10;

(b) $R_+(z) = R_-(z)J_R(z)$ for $z \in \Sigma^R$;

(c) $R(z) = I + O(1/z)$ as $z \to \infty$;

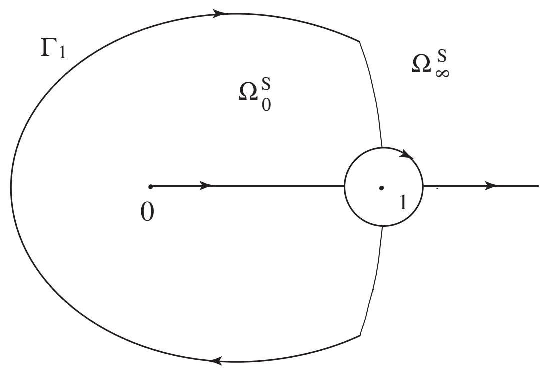

Figure 10: Contour Σ^R for the RH problem for R

(d) $R(z) = O(1)$ as $z \to 1$.

The jump matrices in the above RH problem are given by

$$
\begin{aligned}
J_R(z) &= \begin{pmatrix} 1 & 0 \\ \frac{e^{-\nu\pi i}-e^{\nu\pi i}}{(z-1)^{2b}} n^{-b} e^{n(\phi_{t,n}(z)+\tilde{\phi}_{t,n}(z))} & 1 \end{pmatrix}, & z \in \Gamma_1 \setminus B_\delta, \\
J_R(z) &= \begin{pmatrix} 1 & n^b (z-1)^{2b} e^{-2n\phi_{t,n}(z)} \\ 0 & 1 \end{pmatrix}, & z \in (0, 1-\delta), \\
J_R(z) &= \begin{pmatrix} 1 & n^b |z-1|^{2b} \frac{\sin(\nu+2b)\pi}{\sin \nu\pi} e^{2b\pi i} e^{-2n\tilde{\phi}_{t,n}(z)} \\ 0 & 1 \end{pmatrix}, & z \in (1+\delta, \infty), \\
J_R(z) &= P(z) n^{-\frac{b}{2}\sigma_3}, & z \in \partial B_\delta. \quad (3.75)
\end{aligned}
$$

From Proposition 3.5, the jump matrices are exponentially close to the identity matrix if n is large except for the one on ∂B_δ. For ∂B_δ, we have the following lemma:

Lemma 3.6. *The jump matrix $J_R(z)$ on ∂B_δ is given by*

$$
J_R(z) = P(z) n^{-\frac{b}{2}\sigma_3} = I + \frac{1}{\sqrt{n}} P^{(-1)}(z) + \frac{1}{n} P^{(-2)}(z) + O(n^{-\frac{3}{2}}), \tag{3.76}
$$

uniformly for $z \in \partial B_\delta$, where $P^{(-1)}(z)$ and $P^{(-2)}(z)$ are given by, for $z \in B_\delta$

$$
P^{(-1)}(z) = \frac{1}{f(z)} E^{-1}(z) \sigma_1 \Psi_{-1}\left(L \frac{z-1}{\sqrt{2} f(z)}\right) \sigma_1 E(z) - \frac{\nu\, L(z+1)}{\sqrt{2}\,(z-1)} \sigma_3 \tag{3.77}
$$

and

$$
\begin{aligned}
P^{(-2)}(z) = {} & \frac{1}{f^2(z)} E^{-1}(z) \sigma_1 \Psi_{-2}\left(L \frac{z-1}{\sqrt{2} f(z)}\right) \sigma_1 E(z) + \frac{\nu^2\, L^2 (z+1)^2}{4(z-1)^2} I \\
& + \frac{\nu^2}{(z-1)^2} \sigma_3 - \frac{\nu\, L(z+1)}{\sqrt{2}\,(z-1) f(z)} E^{-1}(z) \sigma_1 \Psi_{-1}\left(L \frac{z-1}{\sqrt{2} f(z)}\right) \sigma_1 E(z) \sigma_3.
\end{aligned} \tag{3.78}
$$

Here $E(z)$ is given by (3.72), Ψ_{-1} and Ψ_{-2} are given in (1.23).

Proof. Since $Q(z) = E(z)\Psi\left(n^{\frac{1}{2}}f(z), L\frac{z-1}{\sqrt{2}f(z)}\right)$, we have from (3.24) and (3.52)

$$P(z) = \sigma_1 E(z)\Psi\left(n^{\frac{1}{2}}f(z), L\frac{z-1}{\sqrt{2}f(z)}\right)\sigma_1 e^{(n(\phi_n(z)+(t-1)\phi_n^0(z))-\frac{1}{2}(\nu+b)\pi i)\sigma_3}(z-1)^{-b\sigma_3} \tag{3.79}$$

for $z \in B_\delta \cap \Omega_0^S$. Using the formulas for $\phi_n^0(z)$ and $\phi_n(z)$ in (3.63)–(3.64), the definition of $E(z)$ in (3.72) and the asymptotic formula for Ψ in (1.23), we get from (3.79)

$$\begin{aligned} P(z)n^{-\frac{b}{2}\sigma_3} =& E^{-1}(z)\Big[I + \frac{\sigma_1\Psi_{-1}\left(L\frac{z-1}{\sqrt{2}f(z)}\right)\sigma_1}{\sqrt{n}f(z)} + \frac{\sigma_1\Psi_{-2}\left(L\frac{z-1}{\sqrt{2}f(z)}\right)\sigma_1}{nf^2(z)} + O(n^{-\frac{3}{2}})\Big] \\ &\times E(z)\ \exp\left[-\frac{\nu\, L(z+1)}{\sqrt{2n}\,(z-1)}\sigma_3 + \frac{\nu^2}{n\,(z-1)^2}\sigma_3 + O(n^{-\frac{3}{2}})\right], \end{aligned} \tag{3.80}$$

which gives us (3.77) and (3.78) for $z \in B_\delta \cap \Omega_0^S$. Similarly, one can prove (3.77) and (3.78) also hold for $z \in B_\delta \cap \Omega_\infty^S$. □

For large n, the jump matrix J_R is close to the identity matrix, both in L^∞ and in L^2-sense on Σ^R. Then following similar analysis as in [9, 33], for large enough n, we know the RH problem for R is solvable. Moreover, from (3.76), $R(z)$ can be expanded as follows

$$R(z) = I + \frac{1}{\sqrt{n}}R^{(-1)}(z) + \frac{1}{n}R^{(-2)}(z) + O(n^{-\frac{3}{2}}), \tag{3.81}$$

as $n \to \infty$, uniformly for $z \in \mathbb{C} \setminus \Sigma^R$. To prove our Theorem 1.1, we need to derive explicit asymptotic formulas for $R^{(-1)}(z)$ and $R^{(-2)}(z)$ as $z \to \infty$.

3.8 Asymptotic formulas for $R^{(-1)}(z)$ and $R^{(-2)}(z)$

To derive asymptotic formulas for $R^{(-1)}(z)$ and $R^{(-2)}(z)$ as $z \to \infty$, more information about $\Psi_{-1}(s)$ and $\Psi_{-2}(s)$ in (1.23) is required. Using Proposition 1.1 in [28, p.51], one can derive the explicit formulas for them as follows:

$$\Psi_{-1}(s) = \begin{pmatrix} -H & -\frac{y}{2} \\ \frac{1}{y}(K - \Theta - \Theta_\infty) & H \end{pmatrix}, \tag{3.82}$$

$$\Psi_{-2}(s) = \begin{pmatrix} (\Psi_{-2}(s))_{11} & (\Psi_{-2}(s))_{12} \\ (\Psi_{-2}(s))_{21} & (\Psi_{-2}(s))_{22} \end{pmatrix}, \tag{3.83}$$

where

$$(\Psi_{-2}(s))_{11} = \frac{1}{2}\Big[H^2 + s\,H - \frac{K}{2} - \frac{1}{2}(\Theta - \Theta_\infty - 1)(\Theta + \Theta_\infty)\Big], \tag{3.84}$$

$$(\Psi_{-2}(s))_{12} = \frac{y}{2}\left(\frac{u}{2} + s - H\right), \tag{3.85}$$

$$(\Psi_{-2}(s))_{21} = \frac{K}{u\,y}(K - 2\Theta) - \frac{s+H}{y}(K - \Theta - \Theta_\infty), \tag{3.86}$$

$$(\Psi_{-2}(s))_{22} = \frac{1}{2}\Big[H^2 - s\,H - \frac{K}{2} + \frac{1}{2}(\Theta - \Theta_\infty + 1)(\Theta + \Theta_\infty)\Big], \tag{3.87}$$

y and K are given in (1.14) and (1.15), respectively, and

$$H(s) = H := \frac{K}{u}(K - 2\Theta) - \left(\frac{u}{2} + s\right)(K - \Theta - \Theta_\infty). \tag{3.88}$$

Then, we obtain the following expansion for $R^{(-1)}(z)$ as $z \to \infty$.

Lemma 3.7. *We have that*

$$R^{(-1)}(z) = \frac{M}{z-1} = \frac{M}{z} + \frac{M}{z^2} + O(z^{-3}) \qquad \text{as } z \to \infty, \tag{3.89}$$

where

$$M := \operatorname*{Res}_{z=1} P^{(-1)}(z) = \sqrt{2}\begin{pmatrix} H(L) - \nu\,L & \frac{1}{y(L)}(K(L) - \nu)\rho^{-2} \\ -\frac{1}{2}\,y(L)\rho^2 & -H(L) + \nu\,L \end{pmatrix} \tag{3.90}$$

and

$$\rho = 2^{-(\nu+b)/2}\,(\nu e^{-1})^{\nu/2}\ e^{\frac{1}{2}(\nu-b)\pi i}. \tag{3.91}$$

Proof. From the RH problem for R and the expansions in (3.76) and (3.81), we know that $R^{(-1)}(z)$ satisfies a RH problem as follows:

(a) $R^{(-1)}(z)$ is analytic for $z \in \mathbb{C} \setminus \partial B_\delta$;

(b) $R_+^{(-1)}(z) = R_-^{(-1)}(z) + P^{(-1)}(z)$ for $z \in \partial B_\delta$;

(c) $R^{(-1)}(z) = O(1/z)$ as $z \to \infty$.

Here ∂B_δ is clockwise oriented. From (3.77) and the property of $f(z)$ in (3.70), we know that $P^{(-1)}(z)$ is analytic in B_δ except for a simple pole at 1. Thus, $R^{(-1)}(z)$ is explicitly given by:

$$R^{(-1)}(z) = \begin{cases} \dfrac{1}{z-1}\operatorname*{Res}_{z=1} P^{(-1)}(z), & z \in \mathbb{C} \setminus \overline{B}_\delta, \\ \dfrac{1}{z-1}\operatorname*{Res}_{z=1} P^{(-1)}(z) - P^{(-1)}(z), & z \in B_\delta. \end{cases} \tag{3.92}$$

This formula immediately gives us (3.89). From (3.77), (3.82) and the definition of $E(z)$ in (3.72), we get (3.90) with ρ given by (3.91). $\square$

Similarly, we get the expansion for $R^{(-2)}(z)$ as $z \to \infty$.

Lemma 3.8. *We have that*

$$R^{(-2)}(z) = \frac{B_{-2}}{(z-1)^2} + \frac{B_{-1}}{z-1} = \frac{B_{-1}}{z} + \frac{B_{-1}+B_{-2}}{z^2} + O(z^{-3}) \qquad \text{as } z \to \infty, \quad (3.93)$$

where B_{-1} is a constant matrix and B_{-2} is given by

$$2\begin{pmatrix} (\Psi_{-2}(L))_{22} - \nu L H(L) + \dfrac{\nu^2(L^2+1)}{2} & \left((\Psi_{-2}(L))_{21} + \frac{\nu L}{y(L)}(K(L)-\nu)\right)\rho^{-2} \\ \left((\Psi_{-2}(L))_{12} + \frac{\nu L}{2} y(L)\right)\rho^2 & (\Psi_{-2}(L))_{11} - \nu L H(L) + \dfrac{\nu^2(L^2-1)}{2} \end{pmatrix} \quad (3.94)$$

with ρ given by (3.91) and $(\Psi_{-2}(s))_{ij}$ given in (3.84)–(3.87).

Proof. As in the proof of Lemma 3.7, we have a RH problem for $R^{(-2)}(z)$ from (3.76) and (3.81):

(a) $R^{(-2)}(z)$ is analytic for $z \in \mathbb{C} \setminus \partial B_\delta$;

(b) $R^{(-2)}_+(z) = R^{(-2)}_-(z) + P^{(-2)}(z) + P^{(-1)}(z)(R^{(-1)})_-(z)$ for $z \in \partial B_\delta$;

(c) $R^{(-2)}(z) = O(1/z)$ as $z \to \infty$.

From (3.70) and (3.78), we see that $P^{(-2)}(z)$ is analytic in B_δ except for a double pole at 1. If we rewrite $P^{(-2)}(z) + P^{(-1)}(z)(R^{(-1)})_-(z)$ as

$$P^{(-2)}(z) + P^{(-1)}(z)(R^{(-1)})_-(z) = \frac{B_{-2}}{(z-1)^2} + \frac{B_{-1}}{z-1} + W(z), \quad (3.95)$$

where B_{-1} and B_{-2} are two constant matrices, $W(z)$ is an analytic function in B_δ. Then, $R^{(-2)}(z)$ can be given as follows

$$R^{(-2)}(z) = \begin{cases} \dfrac{B_{-2}}{(z-1)^2} + \dfrac{B_{-1}}{z-1}, & z \in \mathbb{C} \setminus \overline{B}_\delta, \\ -W(z), & z \in B_\delta. \end{cases} \quad (3.96)$$

Thus, as $z \to \infty$, we have

$$R^{(-2)}(z) = \left(\frac{1}{z^2} + O(z^{-3})\right) B_{-2} + \left(\frac{1}{z} + \frac{1}{z^2} + O(z^{-3})\right) B_{-1}, \quad (3.97)$$

which gives us (3.93). From (3.78), (3.83), (3.95) and the definition of $E(z)$ in (3.72), we have (3.94). $\square$

3.9 Proof of Theorem 1.1 in Case I

Then we are ready for the proof of Theorem 1.1.

Proof. Reversing the transformations $Y \mapsto U \mapsto T \mapsto S \mapsto R$, we can recover Y from R. Since the RH problem for R is solvable for large enough n, it follows that the RH problem Y has a solution for large enough n. This implies that the corresponding monic orthogonal polynomial $\pi_n(z)$ uniquely exists for large enough n.

Next, we are going to calculate the asymptotic formulas for the recurrence coefficients a_n and b_n. From the definition of g_n in (3.20), we have

$$\begin{aligned} g_n(z) &= \int \log(z-s)d\mu_n(s) \\ &= \log z - \frac{1}{z}\int s\ d\mu_n(s) - \frac{1}{2z^2}\int s^2\ d\mu_n(s) + O(z^{-3}) \end{aligned}$$

and so by (3.23)

$$g_{t,n}(z) = g_n(z) + (t-1)g_n^0(z) = \log z - d_1 z^{-1} - d_2 z^{-2} + O(z^{-3}) \qquad \text{as } z \to \infty,$$

for certain constants d_1 and d_2 (that depend on n and t). Then, from (3.33), (3.43) and (3.74), we get

$$\begin{aligned} R(z) &= \Big((-1)^n(e^{-i\pi\nu} - e^{i\pi\nu})n^{-b}\Big)^{-\frac{1}{2}\sigma_3} e^{-\frac{1}{2}nl_{t,n}\sigma_3}\left(I + \frac{Y_{-1}}{z} + \frac{Y_{-2}}{z^2} + O(z^{-3})\right) \\ &\quad \times \left(I + \frac{d_1 n}{z}\sigma_3 + \frac{d_1^2 n^2}{2z^2}I + \frac{d_2 n}{z^2}\sigma_3 + O(z^{-3})\right) e^{\frac{1}{2}nl_{t,n}\sigma_3}\Big((-1)^n(e^{-i\pi\nu} - e^{i\pi\nu})n^{-b}\Big)^{\frac{1}{2}\sigma_3} \\ &= I + \frac{R_{-1}}{z} + \frac{R_{-2}}{z^2} + O(z^{-3}) \end{aligned} \tag{3.98}$$

with

$$R_{-1} = c^{-\sigma_3}\Big(Y_{-1} + d_1 n\sigma_3\Big)c^{\sigma_3}, \tag{3.99}$$

$$R_{-2} = c^{-\sigma_3}\Big(Y_{-2} + d_1 n\, Y_{-1}\sigma_3 + \frac{d_1^2 n^2}{2}I + d_2 n\sigma_3\Big)c^{\sigma_3}, \tag{3.100}$$

where $c = \Big((-1)^n(e^{-i\pi\nu} - e^{i\pi\nu})n^{-b}\Big)^{\frac{1}{2}} e^{\frac{1}{2}nl_{t,n}}$. Recalling (2.4) and (2.5), it immediately follows that

$$a_n = \Big(Y_{-1}\Big)_{12}\Big(Y_{-1}\Big)_{21} = \Big(R_{-1}\Big)_{12}\Big(R_{-1}\Big)_{21} \tag{3.101}$$

and

$$
\begin{aligned}
b_n &= \frac{\Big(Y_{-2}\Big)_{12}}{\Big(Y_{-1}\Big)_{12}} - \Big(Y_{-1}\Big)_{22} = \frac{\Big(R_{-2}\Big)_{12} c^2 + d_1 n \Big(Y_{-1}\Big)_{12}}{\Big(R_{-1}\Big)_{12} c^2} - \Big[\Big(R_{-1}\Big)_{22} + d_1 n\Big] \\
&= \frac{\Big(R_{-2}\Big)_{12} c^2 + d_1 n \Big(R_{-1}\Big)_{12} c^2}{\Big(R_{-1}\Big)_{12} c^2} - \Big[\Big(R_{-1}\Big)_{22} + d_1 n\Big] \\
&= \frac{\Big(R_{-2}\Big)_{12}}{\Big(R_{-1}\Big)_{12}} - \Big(R_{-1}\Big)_{22}. \qquad (3.102)
\end{aligned}
$$

From (3.81), (3.89) and (3.93), we know that

$$
\begin{aligned}
R(z) = I &+ \frac{1}{\sqrt{n}} \left(\frac{1}{z} + \frac{1}{z^2} + O(z^{-3})\right) M \\
&+ \frac{1}{n} \left(\frac{B_{-1}}{z} + \frac{B_{-1} + B_{-2}}{z^2} + O(z^{-3})\right) + O(n^{-\frac{3}{2}}). \qquad (3.103)
\end{aligned}
$$

Combining (3.98) and (3.103), it follows that

$$
R_{-1} = \frac{1}{\sqrt{n}} M + \frac{1}{n} B_{-1} + O(n^{-\frac{3}{2}}) \qquad (3.104)
$$

and

$$
R_{-2} = \frac{1}{\sqrt{n}} M + \frac{1}{n} \Big(B_{-1} + B_{-2}\Big) + O(n^{-\frac{3}{2}}). \qquad (3.105)
$$

Thus, from (3.101), we get

$$
a_n = \frac{1}{n} \Big(M\Big)_{12} \Big(M\Big)_{21} + O(n^{-\frac{3}{2}}). \qquad (3.106)
$$

From (3.90) and the assumption that L is not a pole of $u(s)$ and $K(L) \neq \nu$ in Theorem 1.1, we have $(M)_{12} = \frac{\sqrt{2}}{y(L)}(K(L) - \nu)\rho^{-2} \neq 0$. Then, from (3.102), we have

$$
\begin{aligned}
b_n &= \frac{\Big(M\Big)_{12} + \frac{1}{\sqrt{n}} \Big((B_{-1})_{12} + (B_{-2})_{12}\Big) + O(n^{-1})}{\Big(M\Big)_{12} + \frac{1}{\sqrt{n}} (B_{-1})_{12} + O(n^{-1})} - \frac{1}{\sqrt{n}} \Big(M\Big)_{22} + O(n^{-1}) \\
&= \left[1 + \frac{1}{\sqrt{n}} \frac{(B_{-1})_{12}}{\Big(M\Big)_{12}} + \frac{1}{\sqrt{n}} \frac{(B_{-2})_{12}}{\Big(M\Big)_{12}} + O(n^{-1})\right] \left[1 - \frac{1}{\sqrt{n}} \frac{(B_{-1})_{12}}{\Big(M\Big)_{12}} + O(n^{-1})\right] \\
&\quad - \frac{1}{\sqrt{n}} \Big(M\Big)_{22} + O(n^{-1}) \\
&= 1 + \frac{1}{\sqrt{n}} \frac{(B_{-2})_{12}}{(M)_{12}} - \frac{1}{\sqrt{n}} \Big(M\Big)_{22} + O(n^{-1}). \qquad (3.107)
\end{aligned}
$$

With (3.90) and (3.94), the above formulas give us (1.34) and (1.35).

This completes the proof of Theorem 1.1 for Case I. $\square$

4 Case II: $\nu < 0$

4.1 Introduction

Since $\nu < 0$, with the definition of A_n in (3.2) we know $A_n > 1$. Following the RH analysis in [3], define

$$\beta_n := 2 - A_n + 2i\sqrt{A_n - 1} \tag{4.1}$$

and

$$R_n(z) := \sqrt{(z-\beta_n)(z-\overline{\beta}_n)}, \qquad z \in \mathbb{C} \setminus \Gamma_{0,n}, \tag{4.2}$$

where $R_n(z)$ behaves like z as $z \to \infty$. Recalling the value of A_n in (3.2), β_n can be rewritten as

$$\beta_n = 1 + \frac{\nu}{n} + 2i\sqrt{\frac{-\nu}{n}}. \tag{4.3}$$

As in [3], define

$$\phi_n(z) := \frac{1}{2}\int_{\beta_n}^{z} \frac{R_n(s)}{s} ds, \qquad z \in \mathbb{C} \setminus (\Gamma_{0,n} \cup \Gamma_{1,n} \cup [0,\infty)), \tag{4.4}$$

where the path of integration from β_n to z lies entirely in the region $\mathbb{C} \setminus (\Gamma_{0,n} \cup \Gamma_{1,n} \cup [0,\infty))$, except for the initial point β_n. Similarly define

$$\tilde{\phi}_n(z) := \frac{1}{2}\int_{\bar{\beta}_n}^{z} \frac{R_n(s)}{s} ds, \qquad z \in \mathbb{C} \setminus (\Gamma_{0,n} \cup \Gamma_{2,n} \cup [0,\infty)), \tag{4.5}$$

where the path of integration from $\bar{\beta}_n$ to z lies entirely in the region $\mathbb{C} \setminus (\Gamma_{0,n} \cup \Gamma_{2,n} \cup [0,\infty))$, except for the initial point $\bar{\beta}_n$. It is not difficult to verify (also see [3])

$$\phi_n(z) = \tilde{\phi}_n(z) \mp \pi i \qquad \text{for } z \in \Omega^{\pm}. \tag{4.6}$$

To clarify contours $\Gamma_{0,n}$, $\Gamma_{1,n}$ and $\Gamma_{2,n}$ in (4.4) and (4.5), we consider trajectories of the quadratic differential

$$-\frac{R_n(s)^2}{s^2} ds^2 = -\frac{(s-\beta_n)(s-\bar{\beta}_n)}{s^2} ds^2, \tag{4.7}$$

which has two simple zeros at β_n and $\bar{\beta}_n$ and a double pole at 0; see Strebel [31]. In [3] it is shown that there exist curves $\Gamma_{0,n}$, $\Gamma_{1,n}$ and $\Gamma_{2,n}$ as follows:

Definition 4.1. The contour $\Gamma_{0,n}$ is a curve from $\bar{\beta}_n$ to β_n which crosses the negative real axis, so that

$$\operatorname{Re}\phi_{n,\pm}(z) = 0 \qquad \text{for} \quad z \in \Gamma_{0,n}. \tag{4.8}$$

$\Gamma_{1,n}$ and $\Gamma_{2,n}$ are curves that form the analytic continuation of $\Gamma_{0,n}$ such that $\tilde{\phi}_n(z)$ and $\phi_n(z)$ are real and positive on them, respectively; see Figure 11.

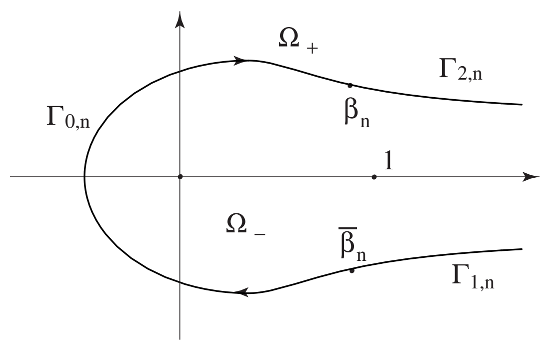

Figure 11: Curves $\Gamma_{0,n}$, $\Gamma_{1,n}$ and $\Gamma_{2,n}$

As in [3], in this case we choose the curve Σ in the RH problem for Y to be $\Sigma = \Gamma_{0,n} \cup \Gamma_{1,n} \cup \Gamma_{2,n}$. And we do not need a step $Y \mapsto U$ as in the Case I.

Following the similar analysis as in the proof of Lemma 3.2, we have:

Lemma 4.2. *As $n \to \infty$, the curve $\Gamma_{0,n}$ tends to the Szegő curve $\mathcal{S}$ (1.5).*

As an analog of (3.11) in the Case I, we define a measure $d\mu_n$ to be

$$d\mu_n(y) = \frac{1}{2\pi i}\frac{R_{n,+}(y)}{y}dy, \qquad y \in \Gamma_{0,n}. \tag{4.9}$$

Using (4.4) and (4.8), one can verify that $d\mu_n(y)$ is a probability measure on $\Gamma_{0,n}$; see also [3].

4.2 The g and ϕ functions

As in the Case I, define the g-function to be

$$g_n(z) := \int_{\Gamma_{0,n}} \log(z-s)d\mu_n(s), \qquad z \in \mathbb{C} \setminus (\Gamma_{0,n} \cup \Gamma_{1,n}), \tag{4.10}$$

where the logarithm $\log(z-s)$ is defined with a cut along $\Gamma_{0,n} \cup \Gamma_{1,n}$. Introduce

$$g_{t,n}(z) := g_n(z) + (t-1)g_n^0(z) \tag{4.11}$$

and

$$\phi_{t,n}(z) := \phi_n(z) + (t-1)\phi_n^0(z), \qquad \tilde{\phi}_{t,n}(z) := \tilde{\phi}_n(z) + (t-1)\phi_n^0(z), \tag{4.12}$$

where

$$g_n^0(z) := \frac{1}{2}(z - R_n(z)) \quad \text{and} \quad \phi_n^0(z) := \frac{R_n(z)}{2}, \qquad z \in \mathbb{C} \setminus \Gamma_{0,n}. \tag{4.13}$$

Note that $g_n^0(z) = O(1/z)$ as $z \to \infty$.

Using similar analysis as in Propositions 3.7 and 3.8 in [3], we get the following properties for $g_{t,n}$ of $\phi_{t,n}$, which is an analog of Proposition 3.4 in the Case I.

Proposition 4.3. *(a) There exists a constant $l_{t,n}$ such that*

$$g_{t,n}(z) = \frac{1}{2}\left(A_n \log z + tz + l_{t,n}\right) - \phi_{t,n}(z), \quad z \in \mathbb{C} \setminus (\Gamma_{0,n} \cup \Gamma_{1,n} \cup [0,\infty)). \tag{4.14}$$

Here $\log z$ *is defined with a cut along* $[0,\infty)$*. And the constant* $l_{t,n}$ *is explicitly given by*

$$l_{t,n} = 2g_{t,n}(\beta_n) - (A_n \log \beta_n + t\beta_n).$$

(b) For $z \in \Sigma$*, we have*

$$g_{t,n,+}(z) - g_{t,n,-}(z) = \begin{cases} 2\pi i, & z \in \Gamma_{1,n}, \\ -2\phi_{t,n,+}(z) = 2\phi_{t,n,-}(z), & z \in \Gamma_{0,n}. \end{cases} \tag{4.15}$$

(c) We also have

$$g_{t,n,+}(z) + g_{t,n,-}(z) = \begin{cases} A_n \log z + tz + l_{t,n}, & z \in \Gamma_{0,n}, \\ A_n \log z + tz + l_{t,n} - 2\tilde{\phi}_{t,n}(z), & z \in \Gamma_{1,n}, \\ A_n \log z + tz + l_{t,n} - 2\phi_{t,n}(z), & z \in \Gamma_{2,n}. \end{cases} \tag{4.16}$$

4.3 First and second transformations $Y \mapsto T$, $T \mapsto S$

To normalize the RH problem at infinity, introduce the first transformation $Y \mapsto T$ to be

$$T(z) = e^{-\frac{1}{2}nl_{t,n}\sigma_3} Y(z)\, e^{-ng_{t,n}(z)\sigma_3}\, e^{\frac{1}{2}nl_{t,n}\sigma_3}. \tag{4.17}$$

From the RH problem for Y in Section 2 and Proposition 4.3, we obtain a RH problem for T as follows.

(a) $T(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma$, $\Sigma = \Gamma_{0,n} \cup \Gamma_{1,n} \cup \Gamma_{2,n}$;

(b)

$$T_+(z) = T_-(z)\begin{pmatrix} e^{2n\phi_{t,n,+}(z)} & (z-1)^{2b} \\ 0 & e^{2n\phi_{t,n,-}(z)} \end{pmatrix} \qquad \text{for } z \in \Gamma_{0,n}; \tag{4.18}$$

$$T_+(z) = T_-(z)\begin{pmatrix} 1 & e^{-2n\tilde{\phi}_{t,n}(z)}(z-1)^{2b} \\ 0 & 1 \end{pmatrix} \qquad \text{for } z \in \Gamma_{1,n}; \tag{4.19}$$

$$T_+(z) = T_-(z)\begin{pmatrix} 1 & e^{-2n\phi_{t,n}(z)}(z-1)^{2b} \\ 0 & 1 \end{pmatrix} \qquad \text{for } z \in \Gamma_{2,n}; \tag{4.20}$$

(c) as $z \to \infty$

$$T(z) = I + O\left(\frac{1}{z}\right), \qquad z \in \mathbb{C} \setminus \Sigma.$$

From (4.8) and (4.12), we know that, when $|t-1|$ is small, the boundary values of $\operatorname{Re}\phi_{t,n}(z)$ on both sides of $\Gamma_{0,n}$ is close to 0. This means the jump matrix for T on $\Gamma_{0,n}$ in (4.18) has oscillatory diagonal entries. Similarly as in the previous case, to remove this oscillating jump, we introduce the second transformation and open the lens as follows. Let $\Gamma_{4,n}$ be a smooth curve connecting β_n, $\bar{\beta}_n$ and 1. Note that since $\operatorname{Im}\beta_n \neq \arg\beta_n$, we can not choose $\Gamma_{4,n}$ to be the curve $\operatorname{Im} z = \arg z$. However, as β_n is close to 1 for n large, we can choose $\Gamma_{4,n}$ such that it is close to the curve $\operatorname{Im} z = \arg z$ for n large. Then choose $\Gamma_{3,n}$ be the continuation of $\Gamma_{4,n}$ such that $\Gamma_{3,n} \cup \Gamma_{4,n}$ becomes a closed contour and contains $\Gamma_{0,n}$ in its interior. Then the curves $\Gamma_{j,n}, j = 0, 1, \cdots, 4$, partition the complex plane into domains Ω_∞ and $\Omega_k, k = 0, 1, 2, 3$; see Figure 12. Define

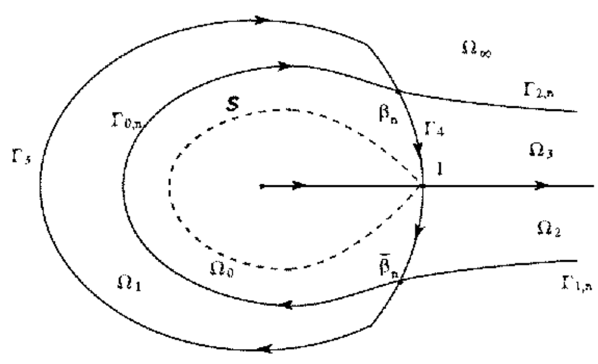

Figure 12: The contour for the RH problem for T

$$S(z) = T(z) \qquad \text{for } z \in \Omega_\infty \tag{4.21}$$

$$S(z) = T(z) \begin{pmatrix} 0 & (z-1)^{2b} \\ -(z-1)^{-2b} & e^{2n\phi_{t,n}(z)} \end{pmatrix} \qquad \text{for } z \in \Omega_0; \tag{4.22}$$

$$S(z) = T(z) \begin{pmatrix} 1 & 0 \\ -(z-1)^{-2b} e^{2n\phi_{t,n}(z)} & 1 \end{pmatrix} \qquad \text{for } z \in \Omega_1; \tag{4.23}$$

$$S(z) = T(z) \begin{pmatrix} 1 & (z-1)^{2b} e^{-2n\tilde{\phi}_{t,n}(z)} \\ 0 & 1 \end{pmatrix} \qquad \text{for } z \in \Omega_2; \tag{4.24}$$

$$S(z) = T(z) \begin{pmatrix} 1 & (z-1)^{2b} e^{-2n\phi_{t,n}(z)} \\ 0 & 1 \end{pmatrix} \qquad \text{for } z \in \Omega_3. \tag{4.25}$$

Direct calculation shows that $S_+(z) = S_-(z)$ for $z \in \Gamma_{0,n} \cup \Gamma_{1,n} \cup \Gamma_{2,n}$. Therefore, $S(z)$ has an analytic continuation across $\Gamma_{0,n} \cup \Gamma_{1,n} \cup \Gamma_{2,n}$ and we obtain a RH problem for S on the contour $\Sigma^S = \Sigma_{3,n} \cup \Sigma_{4,n} \cup [0,\infty)$. Here Σ^S divides the complex plane into two domains Ω_0^S and Ω_∞^S; see Figure 13. As before $(z-1)^{2b}$ is defined with a cut along $[1,\infty)$.

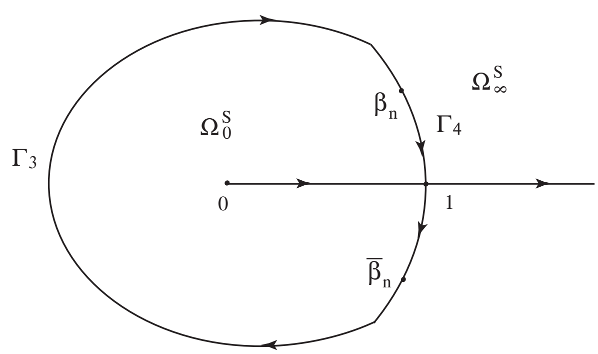

Figure 13: The contour Σ^S for the RH problem for S

(a) S is analytic for $z \in \mathbb{C} \setminus \Sigma^S$, see Figure 13;

(b)

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & 0 \\ (z-1)^{-2b} e^{2n\phi_{t,n}(z)} & 1 \end{pmatrix} \qquad \text{for } z \in \Gamma_{3,n}, \tag{4.26}$$

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & \dfrac{(z-1)^{2b}}{e^{2n\phi_{t,n,+}(z)}} - \dfrac{(z-1)^{2b}}{e^{2n\phi_{t,n,-}(z)}} \\ 0 & 1 \end{pmatrix} \qquad \text{for } z \in (0,1), \tag{4.27}$$

$$S_+(z) = S_-(z) \begin{pmatrix} e^{2n\phi_{t,n}(z)} & 0 \\ (z-1)^{-2b} & e^{-2n\phi_{t,n}(z)} \end{pmatrix} \qquad \text{for } z \in \Gamma_{4,n}, \tag{4.28}$$

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & \dfrac{(z-1)_+^{2b}}{e^{2n\phi_{t,n,+}(z)}} - \dfrac{(z-1)_-^{2b}}{e^{2n\phi_{t,n,-}(z)}} \\ 0 & 1 \end{pmatrix} \qquad \text{for } z \in (1,\infty); \tag{4.29}$$

(c) as $z \to \infty$

$$S(z) = I + O\left(\frac{1}{z}\right);$$

(d) as $z \to 1$

$$S(z) \begin{pmatrix} e^{2n\phi_{t,n,\pm}(1)} & -(z-1)^{2b} \\ (z-1)^{-2b} & 0 \end{pmatrix} = O(1) \quad \text{for } z \in \Omega_0^S \cap \mathbb{C}^\pm, \tag{4.30}$$

$$S(z) \begin{pmatrix} 1 & -e^{-2n\phi_{t,n,+}(1)}(z-1)^{2b} \\ 0 & 1 \end{pmatrix} = O(1) \quad \text{for } z \in \Omega_\infty^S \cap \mathbb{C}^+, \tag{4.31}$$

$$S(z) \begin{pmatrix} 1 & -e^{-2n\tilde{\phi}_{t,n,-}(1)}(z-1)^{2b} \\ 0 & 1 \end{pmatrix} = O(1) \quad \text{for } z \in \Omega_\infty^S \cap \mathbb{C}^-. \tag{4.32}$$

Like in the Case I, condition (d) is important for the above RH problem for S. It makes sure that we have a unique and desired solution for S.

4.4 Definition of ψ functions

For future analysis, it is convenient to introduce new branch cuts for the functions $(z-1)^{2b}$, $R_n(z)$, $\phi_{t,n}(z)$ and $\tilde{\phi}_{t,n}(z)$. More precisely, we want the branch cuts taken on $\Gamma_{4,n}$. Let $\Gamma_{j,+}$ and $\Gamma_{j,-}$ denote $\Gamma_j \cap \mathbb{C}^+$ and $\Gamma_j \cap \mathbb{C}^-$, respectively.

First for $(z-1)^{2b}$, instead of taking the cut along $[1,\infty)$, we take new cut along $\Gamma_{4,n,-} \cup \Gamma_{3,n,-} \cup (-\infty, x_0]$, where x_0 is the intersection point of $\Gamma_{3,n}$ and $(-\infty, 0]$; see Figure 14. Thus,

$$(z-1) \mapsto \begin{cases} (z-1)e^{2\pi i} & \text{for } z \in \Omega_\infty^S \cap \mathbb{C}^-, \\ (z-1) & \text{elsewhere.} \end{cases} \tag{4.33}$$

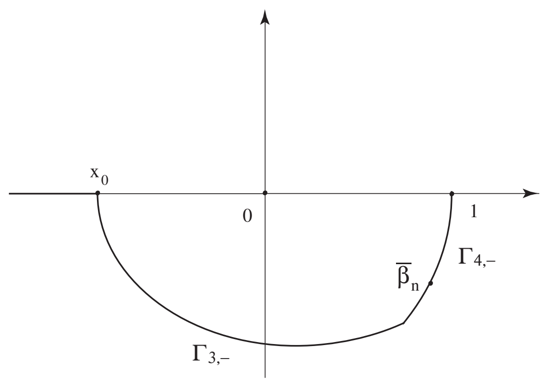

Figure 14: The new cut of $(z-1)^{2b}$

Then, define

$$\widetilde{R}_n(z) := \sqrt{(z-\beta_n)(z-\bar{\beta}_n)}, \qquad z \in \mathbb{C} \setminus \Gamma_{4,n}, \tag{4.34}$$

where $\widetilde{R}_n(z)$ behaves like z as $z \to \infty$. Also introduce

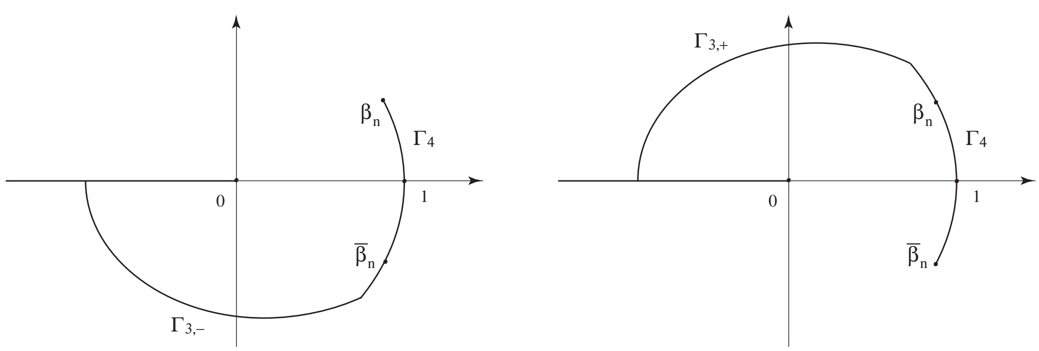

Figure 15: The cuts of $\psi_{t,n}$ and $\tilde{\psi}_{t,n}$

$$\psi_{t,n}(z) := \psi_n(z) + (t-1)\psi_n^0(z) \quad \text{and} \quad \tilde{\psi}_{t,n}(z) := \tilde{\psi}_n(z) + (t-1)\psi_n^0(z) \tag{4.35}$$

where

$$\psi_n(z) := \frac{1}{2}\int_{\beta_n}^{z} \frac{\widetilde{R}_n(s)}{s} ds, \qquad z \in \mathbb{C} \setminus (\Gamma_{3,n,-} \cup \Gamma_{4,n} \cup (-\infty, 0]), \tag{4.36}$$

$$\tilde{\psi}_n(z) := \frac{1}{2}\int_{\bar{\beta}_n}^{z} \frac{\widetilde{R}_n(s)}{s} ds, \qquad z \in \mathbb{C} \setminus (\Gamma_{3,n,+} \cup \Gamma_{4,n} \cup (-\infty, 0]) \tag{4.37}$$

and

$$\psi_n^0(z) := \frac{\widetilde{R}_n(z)}{2}, \qquad z \in \mathbb{C} \setminus \Gamma_{4,n}. \tag{4.38}$$

For illustration of the branch cuts of $\psi_{t,n}$ and $\tilde{\psi}_{t,n}$, see Figure 15. Here are some useful properties of these new auxiliary functions, which can be obtained directly from Proposition 4.6.1 in [5].

Proposition 4.4. *(a) The connection between R_n and $\widetilde{R}_n$:*

$$\widetilde{R}_n(z) = \begin{cases} -R_n(z), & \text{for } z \in \Omega_0, \\ R_n(z), & \text{elsewhere.} \end{cases} \tag{4.39}$$

(b) The connection between $\psi_{t,n}$ and $\phi_{t,n}$:

$$\psi_{t,n}(z) = \begin{cases} \phi_{t,n}(z), & \text{for } z \in (\Omega_3 \cup \Omega_1 \cup \Omega_\infty) \cap \mathbb{C}^+, \\ -\phi_{t,n}(z), & \text{for } z \in \Omega_0 \cap \mathbb{C}^+, \\ -\phi_{t,n}(z) + A_n \pi i, & \text{for } z \in \Omega_0 \cap \mathbb{C}^-, \\ \phi_{t,n}(z) - A_n \pi i, & \text{for } z \in \Omega_2, \\ \phi_{t,n}(z) + A_n \pi i, & \text{for } z \in \Omega_1 \cap \mathbb{C}^-, \\ \phi_{t,n}(z) + (2 - A_n)\pi i, & \text{for } z \in \Omega_\infty \cap \mathbb{C}^-. \end{cases} \tag{4.40}$$

(c) The connection between $\psi_{t,n}$ and $\tilde{\psi}_{t,n}$:

$$\psi_{t,n}(z) = \begin{cases} \tilde{\psi}_{t,n}(z) - \frac{\nu}{n}\pi i, & \text{for } z \in \Omega_0^S, \\ \tilde{\psi}_{t,n}(z) + \frac{\nu}{n}\pi i, & \text{for } z \in \Omega_\infty^S. \end{cases} \tag{4.41}$$

Then, with the help of (4.33) and Proposition 4.4, the RH problem for S can be rewritten as follows. Now $(z-1)^{2b}$ is defined with a cut along $\Gamma_{4,n,-} \cup \Gamma_{3,n,-} \cup (-\infty, x_0]$.

(a) S is analytic for $z \in \mathbb{C} \setminus \Sigma^S$; see Figure 13;

(b)

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & 0 \\ (z-1)^{-2b} e^{2n\psi_{t,n}(z)} & 1 \end{pmatrix}, \qquad z \in \Gamma_{3,n,+}, \tag{4.42}$$

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & 0 \\ (z-1)_-^{-2b} e^{2n\tilde{\psi}_{t,n}(z)} & 1 \end{pmatrix}, \qquad z \in \Gamma_{3,n,-}, \tag{4.43}$$

$$S_+(z) = S_-(z) \begin{pmatrix} e^{2n\psi_{t,n,+}(z)} & 0 \\ (z-1)^{-2b} & e^{2n\psi_{t,n,-}(z)} \end{pmatrix}, \qquad z \in \Gamma_{4,n,+}, \tag{4.44}$$

$$S_+(z) = S_-(z) \begin{pmatrix} e^{2n\tilde{\psi}_{t,n,+}(z)} & 0 \\ (z-1)_-^{-2b} & e^{2n\tilde{\psi}_{t,n,-}(z)} \end{pmatrix}, \qquad z \in \Gamma_{4,n,-}, \tag{4.45}$$

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & (z-1)^{2b} e^{-2n\psi_{t,n}(z)} (e^{-2\nu\pi i} - 1) \\ 0 & 1 \end{pmatrix}, \qquad z \in (0,1), \tag{4.46}$$

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & (z-1)^{2b} e^{-2n\psi_{t,n}(z)} (1 - e^{2(\nu+2b)\pi i}) \\ 0 & 1 \end{pmatrix}, \qquad z \in (1,\infty), \tag{4.47}$$

(c) as $z \to \infty$

$$S(z) = I + O\left(\frac{1}{z}\right);$$

(d) as $z \to 1$

$$S(z) \begin{pmatrix} e^{2n\phi_{t,n,\pm}(1)} & -(z-1)^{2b} \\ (z-1)^{-2b} & 0 \end{pmatrix} = O(1) \quad \text{for } z \in \Omega_0^S \cap \mathbb{C}^\pm, \tag{4.48}$$

$$S(z) \begin{pmatrix} 1 & -e^{-2n\phi_{t,n,+}(1)} (z-1)^{2b} \\ 0 & 1 \end{pmatrix} = O(1) \quad \text{for } z \in \Omega_\infty^S \cap \mathbb{C}^+, \tag{4.49}$$

$$S(z) \begin{pmatrix} 1 & -e^{-2n\tilde{\phi}_{t,n,-}(1) + 4b\pi i} (z-1)^{2b} \\ 0 & 1 \end{pmatrix} = O(1) \quad \text{for } z \in \Omega_\infty^S \cap \mathbb{C}^-. \tag{4.50}$$

Following similar analysis as in the Case I, one can see that most of the jump matrices in the above RH problem tend to the identity matrix as $n \to \infty$ at an exponential rate. More precisely, we have the following proposition.

Proposition 4.5. *Let δ be a fixed small constant and $B_\delta := \{z \in \mathbb{C} \mid |z-1| < \delta\}$. Then there exist $\eta > 0$ and $\varepsilon > 0$ such that for all $t \in \mathbb{R}$ with $|t-1| < \eta$ and for all n large enough, we have*

$$\operatorname{Re} \psi_{t,n}(z) < -\varepsilon \qquad \textit{for } z \in (\Gamma_{3,n} \cup \Gamma_{4,n}) \setminus B_\delta$$

and

$$\operatorname{Re}\psi_{t,n}(z) > \varepsilon(|z|+1) \qquad \textit{for } z \in (0,\infty) \setminus B_\delta.$$

The same inequalities hold for $\operatorname{Re}\tilde{\psi}_{t,n}(z)$.

Proof. This proposition is similar with Proposition 3.5 and the proof is similar, too. □

4.5 Construction of the local parametrix

From Proposition 4.5, we know that the jump matrices for S are exponentially close to the identity matrix as $n \to \infty$, except for the ones in the neighborhood of 1. Let us focus on the RH problem of S restricted to a neighborhood B_δ of $z = 1$. We seek a 2×2 matrix valued function P that satisfies the following RH problem.

(a) $P(z)$ is analytic for $z \in B_\delta \setminus \Sigma^S$, and continuous on $\overline{B}_\delta \setminus \Sigma^S$.

(b) $P(z)$ satisfies the same jump conditions on $\Sigma^S \cap B_\delta$ as S does; see (4.42)–(4.47).

(c) on ∂B_δ, as $n \to \infty$

$$P(z) = \left(I + O\left(\frac{1}{\sqrt{n}}\right)\right) n^{\frac{b}{2}\sigma_3} \qquad \text{for } z \in \partial B_\delta \setminus \Sigma^S. \tag{4.51}$$

(d) $P(z)$ satisfies the same local behavior near 1 as S does; see (4.48)–(4.50).

As in the previous case, we can not avoid the factor $n^{\frac{b}{2}\sigma_3}$ in (4.51).

To transform the RH problem for P into a RH problem for Q with constant jump matrices, we seek P in the following form

$$P(z) = \begin{cases} \sigma_1 Q(z)\sigma_1 e^{(n\psi_{t,n}(z)-\frac{1}{2}(\nu+b)\pi i)\sigma_3}(z-1)^{-b\sigma_3} & \text{for } z \in B_\delta \cap \Omega_0^S, \\ \sigma_1 Q(z)\sigma_1 e^{(n\tilde{\psi}_{t,n}(z)+\frac{1}{2}(\nu-b)\pi i)\sigma_3}(z-1)^{-b\sigma_3} & \text{for } z \in B_\delta \cap \Omega_\infty^S, \end{cases} \tag{4.52}$$

where $(z-1)^b$ is defined with a cut along $\Gamma_{4,n,-} \cup \Gamma_{3,n,-} \cup (-\infty, x_0]$. The factors $e^{n\psi_{t,n}(z)\sigma_3}(z-1)^{-b\sigma_3}$ and $e^{n\tilde{\psi}_{t,n}(z)\sigma_3}(z-1)^{-b\sigma_3}$ in (4.52) are introduced to cancel the corresponding factors in the jump matrices in (4.42)–(4.47) and make them independent of z.

Reorient the curves $\Sigma^S \cap B_\delta$ and let them all extend from the point 1. Then it can be verified that, in order for $P(z)$ to satisfy the desired RH problem, $Q(z)$ should satisfy a RH problem as follows:

(a) $Q(z)$ is analytic for $z \in B_\delta \setminus \Sigma^S$, and continuous on $\overline{B}_\delta \setminus \Sigma^S$;

(b)

$$Q_+(z) = Q_-(z)\begin{pmatrix} 1 & 0 \\ e^{-(\nu+b)\pi i} - e^{(\nu+3b)\pi i} & 1 \end{pmatrix}, \qquad z \in (1, 1+\delta); \quad (4.53)$$

$$Q_+(z) = Q_-(z)\begin{pmatrix} 1 & -e^{(\nu+b)\pi i} \\ 0 & 1 \end{pmatrix}, \qquad z \in (\Gamma_{3,n,+} \cup \Gamma_{4,n,+}) \cap B_\delta; \quad (4.54)$$

$$Q_+(z) = Q_-(z)\begin{pmatrix} 1 & 0 \\ (e^{\nu\pi i} - e^{-\nu\pi i})e^{-(2\nu+b)\pi i} & 1 \end{pmatrix}, \qquad z \in (1-\delta, 1); \quad (4.55)$$

$$Q_+(z) = Q_-(z)\begin{pmatrix} 1 & e^{(3\nu+b)\pi i} \\ 0 & 1 \end{pmatrix} e^{2(\nu+b)\pi i\sigma_3}, \quad z \in (\Gamma_{3,n,-} \cup \Gamma_{4,n,-}) \cap B_\delta; \quad (4.56)$$

(c) for $z \in \partial B_\delta$ as $n \to \infty$,

$$Q(z) = \left(I + O\left(\frac{1}{\sqrt{n}}\right)\right)(\sqrt{n}(z-1))^{-(\nu+b)\sigma_3} \exp\left[\frac{n}{2}(z - 1 - \log z) + \frac{n^{\frac{1}{2}}\sqrt{2}\,L(z-1)}{2} + \frac{\nu}{2}(\log z + \log(-\nu) - 1) - \frac{b}{2}\pi i\right]\sigma_3 \quad (4.57)$$

where $\log z$ and $(z-1)^{-(\nu+b)}$ are defined with cuts along $(-\infty, 0]$ and $\Gamma_{4,n,-} \cup \Gamma_{3,n,-} \cup (-\infty, x_0]$, respectively;

(d) $Q(z)$ has the following behavior as $z \to 1$:

$$Q(z)\begin{pmatrix} 0 & e^{(\nu+b)\pi i} \\ -e^{-(\nu+b)\pi i} & 1 \end{pmatrix}(z-1)^{b\sigma_3} = O(1), \qquad z \in \Omega_0^S \cap \mathbb{C}^+, \quad (4.58)$$

$$Q(z)\begin{pmatrix} 0 & e^{(\nu+b)\pi i} \\ -e^{-(\nu+b)\pi i} & e^{-2\nu\pi i} \end{pmatrix}(z-1)^{b\sigma_3} = O(1), \qquad z \in \Omega_0^S \cap \mathbb{C}^-, \quad (4.59)$$

$$Q(z)\begin{pmatrix} 1 & 0 \\ -e^{-(\nu+b)\pi i} & 1 \end{pmatrix}(z-1)^{b\sigma_3} = O(1), \qquad z \in \Omega_\infty^S \cap \mathbb{C}^+, \quad (4.60)$$

$$Q(z)\begin{pmatrix} 1 & 0 \\ -e^{(\nu+3b)\pi i} & 1 \end{pmatrix}(z-1)^{b\sigma_3} = O(1), \qquad z \in \Omega_\infty^S \cap \mathbb{C}^-, \quad (4.61)$$

where $(z-1)^b$ is defined with a cut along $\Gamma_{4,n,-} \cup \Gamma_{3,n,-} \cup (-\infty, x_0]$.

The above RH problem for Q is very similar to the RH problem in Section 3.6.2. One can see that there is only a constant difference between corresponding formulas in these two RH problems. Also again there is no difference for the jump matrices (4.54) and (4.56) in the neighborhood of β_n and $\bar{\beta}_n$, respectively.

To obtain the limiting behavior as $z \to 1$ in (4.58)–(4.61), one have to make use of (4.6) and Proposition 4.4. To derive the matching condition in (4.57), we need the following asymptotic formulas for $\psi_n^0(z)$ and $\psi_n(z)$ as $n \to \infty$. For $z \in \partial B_\delta$, we have from (4.38)

$$\psi_n^0(z) = \frac{1}{2}(z-1) - \frac{\nu(z+1)}{2n(z-1)} + O\left(\frac{1}{n^2}\right) \tag{4.62}$$

and from (4.36), (calculated with Maple)

$$\begin{aligned}\psi_{1,n}(z) = \frac{1}{2}(z-1-\log z) - \frac{\nu}{2n}\log n + \frac{\nu}{2n}\Big(-2\log(z-1) + \pi i + \log z + \log(-\nu) - 1\Big) \\ + \frac{\nu^2}{n^2(z-1)^2} + O\left(\frac{1}{n^3}\right),\end{aligned} \tag{4.63}$$

where $\log z$ and $\log(z-1)$ are defined with cuts along $(-\infty, 0]$ and $\Gamma_{4,n,-} \cup \Gamma_{3,n,-} \cup (-\infty, x_0]$, respectively; see [5]. The rest of the calculations is the same as in the Case I and we do not go into details.

Now we are in a situation very similar to the Case I in Section 3.6.3. First, from the factors $(z-1)^{b\sigma_3}$ in (4.58)–(4.61) and $(z-1)^{-(\nu+b)\sigma_3}$ in (4.57), again we have

$$\Theta = -b, \qquad \Theta_\infty = \nu + b, \tag{4.64}$$

which are the same as in (3.68). Next, from (4.53)–(4.56) we need the Stokes multipliers

$$\begin{aligned} s_1 &= e^{-(\nu+b)\pi i} - e^{(\nu+3b)\pi i}, & s_2 &= -e^{(\nu+b)\pi i}, \\ s_3 &= (e^{\nu\pi i} - e^{-\nu\pi i})e^{-(2\nu+b)\pi i}, & s_4 &= e^{(3\nu+b)\pi i}. \end{aligned} \tag{4.65}$$

Although (3.67) and (4.65) are not the same, the above Stokes multipliers, like those in (3.67), are again equivalent to those in (1.28) under the transformation in (1.27), but with a different constant $d = -e^{-\nu\pi i}$. Then following almost the same analysis as in the construction of Q in (3.71), except for a constant difference, we construct our parametrix by

$$Q(z) = E(z)\Psi\left(n^{\frac{1}{2}}f(z), L\frac{z-1}{\sqrt{2}f(z)}\right) \tag{4.66}$$

with the same $f(z)$ as given in (3.69) and

$$E(z) = \left(\frac{f(z)}{z-1}\right)^{(\nu+b)\sigma_3} (-\nu e^{-1}\ z)^{\frac{\nu}{2}\sigma_3} e^{-\frac{b}{2}\pi i\sigma_3}. \tag{4.67}$$

There is one thing remaining about the curve $(\Gamma_{3,n} \cup \Gamma_{4,n}) \cap B_\delta$ under the mapping of $f(z)$. Observe that $f(z)$ does not map $(\Gamma_{3,n} \cup \Gamma_{4,n}) \cap B_\delta$ to the imaginary axis. However, for n large enough, according to our choice of $\Gamma_{3,n}$ and $\Gamma_{4,n}$, $f((\Gamma_{3,n} \cup \Gamma_{4,n}) \cap B_\delta)$

is close to the imaginary axis. In the RH problem for Ψ in Section 1.2, by the principle of analytic continuation, we can modify $i\,\mathbb{R}$ to a new contour Λ_n such that $f((\Gamma_{3,n}\cup\Gamma_{4,n})\cap B_\delta)\subset\Lambda_n$ for n large enough. Therefore, we can still use the parametrix constructed in (4.66).

4.6 Proof of Theorem 1.1 in Case II

Having Q as given in (4.66), we can continue the analysis as in Sections 3.7–3.9. The rest of the proof is the same as in the Case I and we do not go into details.

5 Proof of Theorem 1.2

Then we are ready to prove Theorem 1.2.

Proof. Let us consider Case I first. From (2.2), (3.13) and (3.33), we have

$$\pi_n(z) = (Y(z))_{11} = (U(z))_{11} = (T(z))_{11}e^{ng_{t,n}(z)}. \tag{5.1}$$

First for $z\in\Omega_\infty^S$, we get from (3.43)

$$\pi_n(z) = (S(z))_{11}e^{ng_{t,n}(z)}, \qquad z\in\Omega_\infty^S. \tag{5.2}$$

By (3.74) and (3.81), it is known that

$$S(z) = n^{-\frac{b}{2}\sigma_3}R(z)n^{\frac{b}{2}\sigma_3} = n^{-\frac{b}{2}\sigma_3}\left(I+O\left(\frac{1}{\sqrt{n}}\right)\right)n^{\frac{b}{2}\sigma_3} \quad \text{uniformly for } z\in\mathbb{C}\setminus B_\delta. \tag{5.3}$$

Combining (5.2) and (5.3) gives us

$$\pi_n(z) = \left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)e^{ng_{t,n}(z)} \quad \text{uniformly for } z\in\Omega_\infty^S\setminus B_\delta. \tag{5.4}$$

So, for n large enough, there is no zero of $\pi_n(z)$ for $z\in\Omega_\infty^S\setminus B_\delta$.

Then let us consider $z\in\Omega_0^S$. From (3.22) and (3.41) we have

$$(T(z))_{11} = (-1)^n e^{-2n\phi_{t,n}(z)\mp\nu\pi i}(S(z))_{11} + (-1)^n\frac{e^{-\nu\pi i}-e^{\nu\pi i}}{(z-1)^{2b}}(S(z))_{12} \quad \text{for } z\in\Omega_0\cap\mathbb{C}^\pm, \tag{5.5}$$

and from (3.22) and (3.42) we get

$$(T(z))_{11} = (S(z))_{11} + \frac{e^{-\nu\pi i}-e^{\nu\pi i}}{(z-1)^{2b}}e^{2n\phi_{t,n}(z)\pm\nu\pi i}(S(z))_{12} \qquad \text{for } z\in\Omega_1\cap\mathbb{C}^\pm. \tag{5.6}$$

Then, combining (5.1), (5.3), (5.5) and (5.6) gives us

$$\pi_n(z) = e^{ng_{t,n}(z)} \Bigg[(-1)^n e^{-2n\phi_{t,n}(z) \mp \nu\pi i} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right) + (-1)^n \frac{e^{-\nu\pi i} - e^{\nu\pi i}}{n^b (z-1)^{2b}} (R(z))_{12} \Bigg] \quad \text{for } z \in \Omega_0 \cap \mathbb{C}^\pm \setminus B_\delta, \tag{5.7}$$

and

$$\pi_n(z) = e^{ng_{t,n}(z)} \left[1 + O\left(\frac{1}{\sqrt{n}}\right) + \frac{e^{-\nu\pi i} - e^{\nu\pi i}}{n^b (z-1)^{2b}} e^{2n\phi_{t,n}(z) \pm \nu\pi i} O\left(\frac{1}{\sqrt{n}}\right) \right] \quad \text{for } z \in \Omega_1 \cap \mathbb{C}^\pm \setminus B_\delta. \tag{5.8}$$

For $(R(z))_{12}$ in (5.7), we get from (3.81), (3.90) and (3.92)

$$(R(z))_{12} = \frac{\sqrt{2}}{\sqrt{n}(z-1)y(L)} \Big(K(L) - \nu \Big) \rho^{-2} + O(n^{-1}) \quad \text{for } z \in \Omega_0 \setminus B_\delta. \tag{5.9}$$

According to our assumption $K(L) \neq \nu$ in Theorem 1.2, we know that

$$(R(z))_{12} \neq 0 \qquad \text{for } z \in \Omega_0 \setminus B_\delta \text{ and } n \text{ large enough.} \tag{5.10}$$

Let $U(\mathcal{S})$ be the neighborhood of the Szegő curve $\mathcal{S}$. Note that $\Gamma_{0,n}$ tends to $\mathcal{S}$ as $n \to \infty$; see Lemma 3.2. Using the similar analysis as in the proof of Proposition 3.5, it can be shown that there exists $\varepsilon^* > 0$ such that $\operatorname{Re} \phi_{t,n}(z) > \varepsilon^*$ for $z \in \Omega_0 \setminus U(\mathcal{S})$. This means that $e^{-2n\phi_{t,n}(z)}$ is exponentially small as $n \to \infty$. Then, from (5.7) we have

$$\pi_n(z) = e^{ng_{t,n}(z)} (-1)^n \frac{e^{-\nu\pi i} - e^{\nu\pi i}}{(z-1)^{2b}} \left[\frac{\sqrt{2}\Big(K(L) - \nu\Big)}{\sqrt{n}(z-1)y(L)} \rho^{-2} + O(n^{-1}) \right] \tag{5.11}$$

for $z \in \Omega_0 \setminus (B_\delta \cup U(\mathcal{S}))$.. So, for n is large enough, there is no zero of $\pi_n(z)$ in this region.

For $z \in \Omega_1 \setminus U(\mathcal{S})$, similarly it can be shown that $e^{n\phi_{t,n}(z)}$ is exponentially small as $n \to \infty$. Thus, we get from (5.8)

$$\pi_n(z) = \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right) e^{ng_{t,n}(z)}, \quad z \in \Omega_1 \setminus (B_\delta \cup U(\mathcal{S})),$$

which means there is no zero of $\pi_n(z)$. Thus, we can see that all zeros of $\pi_n(z)$ accumulate in $B_\delta \cup U(\mathcal{S})$ as $n \to \infty$.

For Case II, the analysis is similar. This completes the proof of Theorem 1.2. □

6 The solution to PIV with special parameters

It is a well-known fact that for certain parameters Θ and Θ_∞, the PIV equation has special solutions given in terms of parabolic cylinder functions; see e.g. [22, 30, 34, 35, 36, 37]. The special solution of PIV that is of interest in this paper (that is, the one characterized by the Stokes multipliers (1.28)) turns out to be of this type, for certain values of b. We will discuss it briefly in this section.

We recall the RH problem for Ψ from (1.18)–(1.24) with parameters

$$\Theta = -b \quad \text{and} \quad \Theta_\infty = \nu + b, \tag{6.1}$$

and Stokes multipliers (1.28) given in terms of b and ν by

$$\begin{aligned} s_1 &= e^{(2\nu+3b)\pi i} - e^{-b\pi i}, \quad s_2 = e^{b\pi i}, \\ s_3 &= e^{-(2\nu+b)\pi i} - e^{-b\pi i}, \quad s_4 = -e^{(2\nu+b)\pi i}. \end{aligned} \tag{6.2}$$

It turns out that for $b \in \mathbb{Z}/2$, the corresponding special solution of PIV can be expressed in terms of parabolic cylinder functions. If $\nu \in \mathbb{N}$, the parabolic cylinder function reduces to a Hermite function and the PIV solution is a rational function, see also [30, 34, 36, 38].

We discuss the cases $b = 0$ and $b = 1/2$ here.

6.1 The case $b = 0$

Denote by $\Psi^{(b=k/2)}$ the solution of the RH problem for Ψ in Section 1.2 with parameters Θ and Θ_∞ given in (6.1), Stokes multipliers given in (6.2) and $b = \frac{k}{2}$.

When $b = 0$, by (6.1) and (6.2) we have

$$\Theta = 0, \qquad \Theta_\infty = \nu \tag{6.3}$$

and

$$s_1 = e^{2\nu\pi i} - 1, \quad s_2 = 1, \quad s_3 = e^{-2\nu\pi i} - 1, \quad s_4 = -e^{2\nu\pi i}. \tag{6.4}$$

Substituting (6.3) and (6.4) into (1.18)–(1.24), we get the RH problem for $\Psi^{(b=0)}$. For this special case $b = 0$, the RH problem is solved explicitly in terms of parabolic cylinder functions $D_\nu(\zeta)$, where $D_\nu(\zeta)$ is the solution of the second order linear differential equation

$$y''(\zeta) + \left(\nu - \frac{1}{2} - \frac{1}{4}\zeta^2\right) y(\zeta) = 0 \tag{6.5}$$

uniquely characterized by the asymptotic property

$$D_\nu(\zeta) = \zeta^\nu e^{\frac{1}{4}\zeta^2} \left(1 - \frac{\nu(\nu-1)}{2\zeta^2} + O\left(\frac{1}{\zeta^4}\right)\right) \qquad \text{as } \zeta \to \infty, |\arg\zeta| < \frac{3\pi}{4}; \tag{6.6}$$

see [1, Chapter 19]. Other solutions of the differential equation (6.5) are $D_\nu(-\zeta)$ and $D_{-\nu-1}(\pm i\zeta)$. These solutions are related as follows

$$D_\nu(z) = \frac{\Gamma(1+\nu)}{\sqrt{2\pi}}\left(e^{\frac{\nu}{2}\pi i}D_{-\nu-1}(iz) + e^{-\frac{\nu}{2}\pi i}D_{-\nu-1}(-iz)\right), \tag{6.7}$$

$$D_\nu(-z) = \frac{\Gamma(1+\nu)}{\sqrt{2\pi}}\left(e^{-\frac{\nu}{2}\pi i}D_{-\nu-1}(iz) + e^{\frac{\nu}{2}\pi i}D_{-\nu-1}(-iz)\right), \tag{6.8}$$

$$D_\nu(z) = e^{\nu\pi i}D_\nu(-z) - \frac{\Gamma(1+\nu)}{\sqrt{2\pi}}(e^{\nu\pi i} - e^{-\nu\pi i})e^{\frac{\nu}{2}\pi i}D_{-\nu-1}(-iz), \tag{6.9}$$

$$D_\nu(z) = e^{-\nu\pi i}D_\nu(-z) + \frac{\Gamma(1+\nu)}{\sqrt{2\pi}}(e^{\nu\pi i} - e^{-\nu\pi i})e^{-\frac{\nu}{2}\pi i}D_{-\nu-1}(iz); \tag{6.10}$$

see also [39, p.117]. When $\nu \notin \mathbb{Z}$, the solution to the RH problem for $\Psi^{(b=0)}$ is given as follows, which can be verified with (6.6)–(6.10),

$$\Psi^{(b=0)}(l) = \begin{cases} \left(2^{-\nu}e^{s^2}(e^{2\nu\pi i}-1)\right)^{-\frac{\sigma_3}{2}} V(l)\,(e^{2\nu\pi i}-1)^{\frac{\sigma_3}{2}}, & \operatorname{Re} l > 0, \\ \left(2^{-\nu}e^{s^2}(1-e^{-2\nu\pi i})\right)^{-\frac{\sigma_3}{2}} V(l)\,(1-e^{-2\nu\pi i})^{\frac{\sigma_3}{2}}, & \operatorname{Re} l < 0, \end{cases} \tag{6.11}$$

where

$$V(l) = \begin{cases} \begin{pmatrix} e^{-\frac{\nu}{2}\pi i}D_{-\nu}(-i\sqrt{2}(l+s)) & -\frac{\sqrt{2\pi}\,i}{\Gamma(\nu)}D_{\nu-1}(\sqrt{2}(l+s)) \\ \frac{\Gamma(1+\nu)}{\sqrt{2\pi}}e^{-\frac{\nu}{2}\pi i}D_{-\nu-1}(-i\sqrt{2}(l+s)) & D_\nu(\sqrt{2}(l+s)) \end{pmatrix} & \text{for } \arg l \in (0, \frac{\pi}{2}), \\ \begin{pmatrix} e^{\frac{\nu}{2}\pi i}D_{-\nu}(-i\sqrt{2}(l+s)) & \frac{\sqrt{2\pi}\,i}{\Gamma(\nu)}D_{\nu-1}(-\sqrt{2}(l+s)) \\ \frac{\Gamma(1+\nu)}{\sqrt{2\pi}}e^{\frac{\nu}{2}\pi i}D_{-\nu-1}(-i\sqrt{2}(l+s)) & D_\nu(-\sqrt{2}(l+s)) \end{pmatrix} & \text{for } \arg l \in (\frac{\pi}{2}, \pi), \\ \begin{pmatrix} e^{-\frac{\nu}{2}\pi i}D_{-\nu}(i\sqrt{2}(l+s)) & \frac{\sqrt{2\pi}\,i}{\Gamma(\nu)}D_{\nu-1}(-\sqrt{2}(l+s)) \\ -\frac{\Gamma(1+\nu)}{\sqrt{2\pi}}e^{-\frac{\nu}{2}\pi i}D_{-\nu-1}(i\sqrt{2}(l+s)) & D_\nu(-\sqrt{2}(l+s)) \end{pmatrix} & \text{for } \arg l \in (\pi, \frac{3\pi}{2}), \\ \begin{pmatrix} e^{\frac{\nu}{2}\pi i}D_{-\nu}(i\sqrt{2}(l+s)) & -\frac{\sqrt{2\pi}\,i}{\Gamma(\nu)}D_{\nu-1}(\sqrt{2}(l+s)) \\ -\frac{\Gamma(1+\nu)}{\sqrt{2\pi}}e^{\frac{\nu}{2}\pi i}D_{-\nu-1}(i\sqrt{2}(l+s)) & D_\nu(\sqrt{2}(l+s)) \end{pmatrix} & \text{for } \arg l \in (\frac{3\pi}{2}, 2\pi); \end{cases} \tag{6.12}$$

see also [5, 6]. Then, by (6.11) and (6.12), we have

$$(\Psi^{(b=0)}(l))_{12} = -2^{\frac{\nu}{2}}e^{-\frac{1}{2}s^2}\frac{\sqrt{2\pi}\,i}{\Gamma(\nu)\,(e^{2\nu\pi i}-1)}D_{\nu-1}(\sqrt{2}(l+s)) \quad \text{for } \operatorname{Re} l > 0. \tag{6.13}$$

Recalling (3.82) and (6.6), we get from above formula

$$\begin{aligned} y^{(b=0)}(s) &= 2\lim_{l\to\infty} l\, 2^{\frac{\nu}{2}} e^{-\frac{1}{2}s^2} \frac{\sqrt{2\pi}\, i}{\Gamma(\nu)\,(e^{2\nu\pi i}-1)} D_{\nu-1}(\sqrt{2}(l+s)) e^{\frac{l^2}{2}+sl} l^{-\nu} \\ &= c_0\, e^{-s^2}, \end{aligned} \tag{6.14}$$

where

$$c_0 = \frac{2^{\nu+1}\sqrt{\pi}\, i}{\Gamma(\nu)\,(e^{2\nu\pi i}-1)}. \tag{6.15}$$

Furthermore, by (1.14) we get

$$u^{(b=0)}(s) = -2s - \frac{d}{ds}\log y^{(b=0)}(s) \equiv 0, \tag{6.16}$$

which is the trivial solution of the PIV equation with parameter $\Theta = 0$.

It is easily seen that, with (6.16), the expressions (1.34) and (1.35) for the recurrence coefficients of the generalized Laguerre polynomials indeed reduce to (1.9).

6.2 The case $b = \frac{1}{2}$

Knowing the precise form of $\Psi^{(b=0)}(l)$, we are going to compute $\Psi^{(b=1/2)}(l)$. For $b = \frac{1}{2}$, by (6.1) and (6.2) we have

$$\Theta = -\frac{1}{2}, \qquad \Theta_\infty = \nu + \frac{1}{2} \tag{6.17}$$

and

$$s_1 = -i\, e^{2\nu\pi i} + i, \quad s_2 = i, \quad s_3 = -i\, e^{-2\nu\pi i} + i, \quad s_4 = -i\, e^{2\nu\pi i}. \tag{6.18}$$

Define

$$\Phi(l) := e^{-\frac{1}{4}\pi i\sigma_3} \Psi^{(b=1/2)}(l) e^{\frac{1}{4}\pi i \sigma_3} (\Psi^{(b=0)}(l))^{-1}. \tag{6.19}$$

From the RH problems for $\Psi^{(b=0)}$ and $\Psi^{(b=1/2)}$, with $l^{1/2}$ defined with a cut along $i\,\mathbb{R}_-$, we easily obtain the RH problem for $l^{1/2}\Phi(l)$ as follows:

(a) $l^{1/2}\Phi(l)$ is analytic for $l \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$; see Figure 3;

(b) $\left(l^{1/2}\Phi(l)\right)_+ = \left(l^{1/2}\Phi(l)\right)_-$ for $l \in \mathbb{R} \cup i\,\mathbb{R}$;

(c) as $l \to \infty$

$$l^{1/2}\Phi(l) = \begin{pmatrix} 1 & -i\,(\Psi_{-1}^{(b=1/2)})_{12} \\ -(\Psi_{-1}^{(b=0)})_{21} & l + (\Psi_{-1}^{(b=1/2)})_{22} - (\Psi_{-1}^{(b=0)})_{22} \end{pmatrix} + O(l^{-1}), \tag{6.20}$$

where $\Psi_{-1}(s)$ is given in (3.82);

(d) as $l \to 0$

$$l^{1/2}\Phi(l) = O(1). \tag{6.21}$$

From parts (b) and (d) of the RH problem it follows that $l^{1/2}\Phi(l)$ is an entire function in the complex λ plane. By (6.20) we then have for $l \in \mathbb{C}$,

$$l^{1/2}\Phi(l) = \begin{pmatrix} 1 & -i\,(\Psi_{-1}^{(b=1/2)})_{12} \\ -(\Psi_{-1}^{(b=0)})_{21} & l + (\Psi_{-1}^{(b=1/2)})_{22} - (\Psi_{-1}^{(b=0)})_{22} \end{pmatrix} \tag{6.22}$$

Recalling $(\Psi_{-1}^{(b=1/2)})_{12} = -\frac{1}{2}\, y^{(b=1/2)}$ (see (3.82)), we obtain

$$l^{1/2}\Phi(l) = \begin{pmatrix} 1 & \frac{i}{2}\, y^{(b=1/2)} \\ * & * \end{pmatrix}, \tag{6.23}$$

where $*$ denotes entries that are not important for what follows. By (6.19), we have

$$l^{1/2}\Psi^{(b=1/2)}(l) = e^{\frac{1}{4}\pi i\sigma_3} l^{1/2}\Phi(l)\Psi^{(b=0)}(l)e^{-\frac{1}{4}\pi i\sigma_3}. \tag{6.24}$$

Combining (6.11), (6.12) and (6.23), we get that the $(1,2)$ entry of $l^{1/2}\Psi^{(b=1/2)}(l)$ for l in the fourth quadrant is equal to

$$2^{\frac{\nu}{2}} e^{-\frac{1}{2}s^2} \frac{\sqrt{2\pi}}{\Gamma(\nu)\,(e^{2\nu\pi i}-1)} D_{\nu-1}(\sqrt{2}(l+s)) - \frac{y^{(b=1/2)}(s)}{2} 2^{-\frac{\nu}{2}} e^{\frac{1}{2}s^2} D_\nu(\sqrt{2}(l+s)). \tag{6.25}$$

From (1.24), we know

$$l^{1/2}\Psi^{(b=1/2)}(l) = O\begin{pmatrix} 1 & l \\ 1 & l \end{pmatrix} \qquad \text{as } l \to 0 \text{ and } l \text{ in the fourth quadrant.} \tag{6.26}$$

This, together with (6.25), gives us

$$2^{\frac{\nu}{2}} e^{-\frac{1}{2}s^2} \frac{\sqrt{2\pi}}{\Gamma(\nu)\,(e^{2\nu\pi i}-1)} D_{\nu-1}(\sqrt{2}s) - \frac{y^{(b=1/2)}(s)}{2} 2^{-\frac{\nu}{2}} e^{\frac{1}{2}s^2} D_\nu(\sqrt{2}s) = 0, \tag{6.27}$$

so that

$$y^{(b=1/2)}(s) = \frac{c_{1/2}\; D_{\nu-1}(\sqrt{2}\,s)}{e^{s^2} D_\nu(\sqrt{2}\,s)}, \qquad c_{1/2} = \frac{2^{\nu+3/2}\sqrt{\pi}}{\Gamma(\nu)\,(e^{2\nu\pi i}-1)}. \tag{6.28}$$

From (1.14), we then see that the relevant solution of PIV in case $b = 1/2$ is equal to

$$\begin{aligned} u^{(b=1/2)}(s) &= -2s - \frac{d}{ds}\log y^{(b=1/2)}(s) \\ &= \frac{d}{ds}\log \frac{D_\nu(\sqrt{2}\,s)}{D_{\nu-1}(\sqrt{2}\,s)}. \end{aligned} \tag{6.29}$$

From this formula, it is readily seen that the poles of $u^{(b=1/2)}(s)$ are the zeros of $D_\nu(\sqrt{2}\,s)$ and $D_{\nu-1}(\sqrt{2}\,s)$. For ν is real, the parabolic cylinder function D_ν has $\max(\lceil \nu \rceil, 0)$ zeros on the real axis, where $\lceil \cdot \rceil$ is the ceiling function; see [39, p.126]. So, the PIV solution $u^{(b=1/2)}$ has poles on the real axis if $\nu > 0$.

6.3 The case $b = \frac{k}{2}$, $k \in \mathbb{Z}$

Repeating the previous calculations, it is possible (at least in principle) to find the solution $u^{(b=k/2)}(s)$ for PIV with $k \in \mathbb{Z}$ in terms of parabolic cylinder functions. For example, for $b = 1$ we have

$$y^{(b=1)}(s) = c_1 \frac{\mathcal{W}\Big(D_{\nu-1}(\sqrt{2}\,s), D_\nu(\sqrt{2}\,s)\Big)}{e^{s^2}\,\mathcal{W}\Big(D_\nu(\sqrt{2}\,s), D_{\nu+1}(\sqrt{2}\,s)\Big)}, \tag{6.30}$$

where $\mathcal{W}$ is the Wronskian with respect to s and

$$c_1 = -\frac{2^{\nu+2}\sqrt{\pi}\,i}{\Gamma(\nu)\,(e^{2\nu\pi i} - 1)}. \tag{6.31}$$

Thus,

$$u^{(b=1)}(s) = \frac{d}{ds} \log \frac{\mathcal{W}\Big(D_\nu(\sqrt{2}\,s), D_{\nu+1}(\sqrt{2}\,s)\Big)}{\mathcal{W}\Big(D_{\nu-1}(\sqrt{2}\,s), D_\nu(\sqrt{2}\,s)\Big)}. \tag{6.32}$$

Again we see that there are poles on the real axis if $\nu > 0$.

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